Plasma decontamination and energy transport by impurity driven modes

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A radial flow of impurities toward the outer edge of a magnetically confined plasma is produced, following an accumulation of impurities at the center of the plasma column, by impurity density gradient driven modes for realistic values of the temperature gradient of the main ion population relative to its density gradient. At the same time an outward transport of the main ion thermal energy is also induced. For mean free paths of the main ion population shorter than the relevant longitudinal wavelengths, these modes can be described by moment equations and are associated with the finite thermal conductivity of the main ions. In the opposite (collisionless) limit, mode-particle resonances replace the effects of finite ion thermal conductivity and in addition, modes that are standing along the magnetic field lines are affected by the magnetic trapping of a fraction of the main ion population. The expression for their growth rate is strongly influenced by the ratio of the effective trapped ion collision frequency to the mode frequency and under realistic conditions, can be independent of mode-particle resonance processes that involve only a relatively small portion of velocity space.

I. INTRODUCTION

A number of recent high-temperature toroidal plasma experiments have made it clear that the presence of impurities can have significant effects on the equilibrium state and transport properties of the plasmas involved. In the present paper we examine impurity driven plasma modes in a number of different collisionality and wavelength regimes, and estimate the quasi-linear effects that the modes can have on the transport of impurity ions across the magnetic field[1] and on the related ion thermal energy transport.

In Sec. II, we distinguish the different regimes of interest: a collisional regime when the main ion collisional mean free path is shorter than the distance over which the magnetic field is periodically modulated and a trapped ion regime when it is much longer. Also, we distinguish a regime with modes described by fluid equations when the mean free path is shorter than the wavelength of the mode along the magnetic field, and a regime with modes described by kinetic equations involving consideration of velocity space in the opposite limit. Finally, we distinguish between travelling modes with wavelengths along the magnetic field much shorter than the magnetic field periodicity length, and standing modes with modulation lengths of the same order as that of the magnetic field.

In Sec. III, we consider dissipative fluid modes[2,3] in two opposite limits of the main-ion impurity-ion collision frequency, which are associated with the finite longitudinal thermal conductivity of the main ion population and the thermoelectric effect, and depend on friction due to main-ion impurity-ion collisions. The mode corresponding to large collision frequency can be unstable for realistic values and signs of the main ion and impurity ion density gradients and of the main ion temperature gradient.

In Sec. IV, we consider kinetic modes with wavelengths along the magnetic field short compared with both the magnetic field periodicity length and the collisional mean free path.[4] We distinguish an "impurity-sound mode" and an "impurity drift mode" for values of the impurity strength, or effective plasma ion charge number, considerably larger than or of the order of unity, respectively, and impurity ion relative temperature gradients of the same order as the relative impurity ion density gradient. For large impurity ion relative temperature gradients, we find a fluid-like (nonresonant) instability. When the impurity ion density gradient is much larger in magnitude than that of the main ions, fluid-like instabilities can also occur.

The impurity particle flux and the main ion thermal energy flux are calculated for all these modes in the quasi-linear approximation, in terms of the amplitude of the perturbed electrostatic potential. A sequence of instabilities is envisioned to be excited at different stages of the evolution of the impurity density radial distribution. The result of this sequence is that impurities are carried outward,[1] in the presence of a finite temperature gradient relative to the density gradient of the main ions, and become accumulated at the outer edge of the plasma column, a process that we indicate as "self-decontamination."

In Sec. V we study collisionless modes that are standing along the magnetic field lines in the collisionless (Vlasov) limit. The modulation periodicity of these modes is that of the magnetic field and the relevant perturbed electrostatic potentials are odd around the point of minimum magnetic field along a given field line. A general quadratic form, in the perturbed electrostatic potential, of the quasi-neutrality condition, which can be regarded as an effective dispersion relation, is derived, using fluid equations for the impurity ions and the guiding center approximation for the main ions. On the basis of this quadratic form we identify an impurity sound mode and an impurity trapped particle mode corresponding to different values of the impurity strength. For a main ion temperature gradient smaller than a critical value, instability requires that the main ion and impurity ion density gradients be in opposite directions, while for larger main ion temperature gradients, instability can occur with the density gradients in the same direction. The impurity trapped particle mode can be seen to become macroscopically unstable, in the sense of not depending on wave-particle resonances which involve only a small portion of velocity space, under realistic conditions of the various density and temperature gradients involved.

The same formalism that leads to the quadratic form, in the perturbed electric potential mentioned here is used to derive the quasi-linear equations for the various average distribution functions. On this basis, estimates of the impurity particle flux and the main ion thermal energy flux are made.

In Sec. VI, the effects of main ion collisions are included, specifically for trapped main ion effective collision frequencies less than the average trapped ion bounce frequency, by means of a simplified collision operator. With respect to odd modes, whose perturbed electrostatic potential is antisymmetric about the point of minimum magnetic field, we find a dissipative mode whose growth rate depends on the collision frequency and which can be identified with the drift mode discussed in Sec. V.

The effect of impurities on even modes, whose perturbed electrostatic potential is symmetric around the point of minimum magnetic field, is illustrated by the dissipative trapped ion mode that was treated in Ref. 5. Here, we find that in the presence of a finite ion temperature gradient, impurities can stabilize this mode only if they have a reverse and much stronger relative density gradient.

In Sec. VII, we recall the conditions for the equilibrium collisional (classical) impurity particle flux to vanish in various regimes of the main ions and the impurity ions. In general, this condition occurs when the impurities are accumulated at the center of the plasma column for main ion temperature gradients less than a certain critical value, and when the impurities are concentrated at the outside of the plasma column for main ion temperature gradients greater than this critical value. This collisional critical temperature gradient is always greater than the corresponding critical temperature gradient for the impurity modes considered in Secs. III, IV, and VI. Thus, for values of the actual ion temperature gradient between the two critical temperature gradients, we can have a situation in which the impurities are concentrated at the outside of the plasma, with the classical inward impurity flux balanced by the outward impurity flux due to the various impurity-driven modes. Some of the properties of these modes are summarized in Table I.

II. REGIMES OF INTEREST

Consider a plasma that is confined by a magnetic field B that is periodically modulated in magnitude over a distance L. For simplicity, this plasma is assumed to be composed of three species: electrons, main ions with charge number $Z_i=1$, and impurity ions with charge number $Z \gg 1$. The relevant quantities are indicated by the subscripts e, i, and I, respectively. We recall that a quantity of frequent use in this connection is

$$Z^E=(n_I Z^2+n_i)/n_e\ , \tag{1}$$

where $n_e=n_I Z+n_i$, so that $Z^E \gtrsim 1$. We shall mostly consider the realistic case where $Z^E \sim 1$. We indicate the collision mean free path of the main ion population by λ_i and consider

(i) a *collisional regime* where

$$\lambda_i < L\ , \tag{2}$$

and

(ii) a *trapped ion regime* where

$$\lambda_i(\Delta B/B)^{3/2} > L\ , \tag{3}$$

ΔB representing the magnetic field modulation. We recall that $\lambda_i=v_i/\nu_i$, $v_i=(2T_i/m_i)^{1/2}$, and

$$\nu_i=\nu_{ii}+\nu_{iI}\ , \quad \nu_{iI}/\nu_{ii}\sim Z^2 n_I/n_i\ , \tag{4}$$

ν_{ii} and ν_{iI} being the average ion–ion and ion–impurity collision frequencies, respectively. In the case of the impurities

$$\nu_I=\nu_{Ii}+\nu_{II}\ , \quad \frac{\nu_{Ii}}{\nu_{iI}}=\frac{m_i}{m_I}\frac{n_i}{n_I}\ ,$$
$$\frac{\nu_{II}}{\nu_{ii}}\sim Z^4\frac{n_I}{n_i}\left(\frac{T_i}{T_I}\right)^{3/2}\left(\frac{m_i}{m_I}\right)^{1/2}\ . \tag{5}$$

We shall consider

$$T_I \sim T_i \lesssim T_e \tag{6}$$

as a realistic ordering.

We refer to low-β plasmas, with $8\pi(n_iT_i+n_eT_e)\ll B^2$, and limit consideration to electrostatic modes such that

$$\tilde{\mathbf{E}}=-\nabla\tilde{\Phi}\ , \quad \tilde{\Phi}=\tilde{\Phi}(\mathbf{x})\,e^{-i\omega t}\ . \tag{7}$$

In the case of a one-dimensional, plane equilibrium configuration in which the confining magnetic field is in the z direction, we may take

$$\tilde{\Phi}(\mathbf{x})=\tilde{\phi}(x)\exp(i k_y y+ik_\parallel z)\ , \tag{8}$$

and the relevant longitudinal wavelength is

$$\lambda_\parallel=2\pi/k_\parallel\ . \tag{9}$$

In the case of more complex geometry, we take, for order of magnitude estimates,

$$\lambda_\parallel \sim \left|\frac{\mathbf{B}\cdot\nabla\tilde{\Phi}(\mathbf{x})}{B\tilde{\Phi}(\mathbf{x})}\right|^{-1}\ . \tag{10}$$

In particular, if we consider modes with

TABLE I. Impurity driven modes.

Regime[a]	$\lambda_i < \lambda_\parallel < L$	$\lambda_i < \lambda_\parallel < L$	$\lambda_\parallel < L; \lambda_\parallel < \lambda_i$	$\lambda_\parallel \sim L; \lambda_\parallel < \lambda_i$	$\lambda_\parallel \sim L; \lambda_\parallel < \lambda_i$	$\lambda \sim L; \lambda \ll \lambda_i$
Label	Collisional fluid mode I[b]	Collisional fluid mode II[b]	Resonant "drift" mode	Impurity sound mode	Dissipative impurity mode	Collisionless trapped-ion mode
Frequency range	$v_I < \omega/k_\parallel < v_i < v_e$	$v_I < \omega/k_\parallel < v_i < v_e$	$v_I < \omega/k_\parallel < v_i < v_e$	$v_I < \omega/k_\parallel < v_i < v_e$	$\langle\omega_t\rangle_I < \omega < \langle\omega_b\rangle_i < \langle\omega_b\rangle_e$	$\langle\omega_t\rangle_I < \omega < \langle\omega_b\rangle_i < \langle\omega_b\rangle_e$
Growth rate[c]	Eq. (41) nonresonant	Eq. (71) nonresonant	Eq. (83) resonant	Eq. (88)[d] resonant	Eq. (195) nonresonant	Eq. (154) or (156) nonresonant or resonant
Impurity flow[e]	Outward[f] for $\eta_i > \eta_{ic}$ and $\sigma_I > 0$. Inward for $\eta_i < \eta_{ic}$ and $\sigma_I < 0$.	Outward for $\eta_i > \frac{2}{3}$ and $\sigma_I > 0$. Inward for $\eta_i < \frac{2}{3}$ and $\sigma_I < 0$.	Outward[g] for $\sigma_I > 0$ and $\eta_i > \eta_{ic}$. Inward for $\sigma_I > 0$ and $\eta_i < \eta_{ic}$.	Outward[d,g] for $\eta_i > \eta_{ic}$ Inward for $\eta_i < \eta_{ic}$.	Outward for $\sigma_I > 0$ and $\eta_i > \frac{2}{3}$. Inward for $\sigma_i < 0$ and $\eta_i < \frac{2}{3}$.	Outward[h] for $\sigma_I > 0$ and $\eta_i > \eta_{ic}$. Inward for $\sigma_I < 0$ and $\eta_i < \eta_{ic}$.
Main ion thermal energy flow[e]	Outward[i]	Outward[i]	Outward[i]	Outward[d]	Outward[i]	Outward[i]

[a]Here, λ_i is the main ion collisional mean free path, $\lambda_\parallel$ is the mode wavelength along the magnetic field, and L is the magnetic field periodicity length.

[b]Here, mode I (II) corresponds to $\langle\nu_{iI}/\omega\rangle > (<) \; (m_I n_I/m_i n_i)$, where ν_{iI} is the collision frequency of the main ions with the impurities.

[c]Here, "resonant" means the growth rate is due to a mode-particle resonance term involving only a small portion of velocity space.

[d]This mode requires $Z_e \equiv Z^2 n_I/n_i \gtrsim 5(T_I/T_i)$ for instability.

[e]Here, $\eta_i \equiv (d \ln T_i/dr)/(d \ln n_i/dr)$, $r_{nI} \equiv -(d \ln n_I/dr)^{-1}$, $\sigma_I \equiv (d \ln n_I/dr)/(d \ln_i/dr)$, and $Z_e \equiv Z^2 n_I/n_i$.

[f]Here, $\eta_{ic} = \frac{2}{3}\{1 + (Z_e/Z)(n_i/n_e)\sigma_I/[1 + n_I T_I/(n_e T_e)]\}$ for $|\sigma_I| \sim Z > 1$.

[g]Here, $\eta_{ic} \simeq 2(1 - \omega/\omega_{*i})/(1 + 2b_i)$ for $b_i < 1$ and $\eta_{ic} \simeq (1 - \omega/\omega_{*i})$ for $b_i > 1$, where $b_i \equiv \frac{1}{2} k_\perp^2 \rho_i^2$, ρ_i is the main ion gyroradius, and $k_\perp \simeq k_y$ is the mode wavenumber perpendicular to the magnetic field.

[h]Normally, $\eta_{ic} \simeq \frac{2}{3}$. A general expression for η_{ic} is given by Eq. (166).

[i]Whenever the mode is unstable, that is for $\eta_i > \eta_{ic}$ and $\sigma_I > 0$ or for $\eta_i < \eta_{ic}$ and $\sigma_I < 0$.

$$\lambda_\parallel \ll L \; , \tag{11}$$

we may ignore the effects of the equilibrium magnetic field periodicity and refer to a one-dimensional, plane equilibrium as an adequate representation of more complex confinement configurations. The relevant modes can be represented by Eq. (8) and they are of the *traveling* type.

In the case where

$$\lambda_\parallel \sim L \; , \tag{12}$$

it is important to take into account the effects of the magnetic field modulation, such as particle trapping, and refer to a realistic equilibrium configuration. In particular, the relevant modes may be *standing* along the magnetic field and will be described in Secs. V and VI.

Referring to the length of the ion mean free path, we have *collisional fluid modes* if

$$\lambda_i < \lambda_\parallel \; , \tag{13}$$

and in this case moment equations with the appropriate transport coefficients give an adequate description. On the other hand, when

$$\lambda_\parallel < \lambda_i \; , \tag{14}$$

the fluid description is inadequate and appropriate kinetic equations have to be adopted. In particular, when

$$\lambda_\parallel \sim L < \lambda_i (\Delta B/B)^{3/2} \; , \tag{15}$$

trapped ions have a strong influence on the stability of impurity driven modes. We shall label them as either dissipative or collisionless, depending on whether their growth rate depends on the collision frequency of trapped ions or not.

III. DISSIPATIVE FLUID MODES

As was pointed out in Sec. II, collisional fluid modes[2,3,6] described by moment equations[7] can be found for

$$\lambda_i < \lambda_\parallel < L \; . \tag{16}$$

We neglect all finite gyroradius effects and adopt a guiding center description for all species. The modes of interest are associated with the finite thermal conductivity, along the magnetic field, of the main ion population, the thermo-electric effect, and the friction due to i-I collisions. They are found in the frequency range

$$v_I < \omega/k_\parallel \lesssim v_i < v_e \; . \tag{17}$$

The relevant range of mean free paths is such that

$$\frac{k_\parallel^2 v_I^2}{\nu_I} < \omega \lesssim \frac{k_\parallel^2 v_i^2}{\nu_i} < \frac{k_\parallel^2 v_e^2}{\nu_e} \; , \tag{18}$$

ν_j representing the average collision frequency for each species.

The modes under consideration satisfy the quasi-neutrality condition

$$\tilde{n}_e = \tilde{n}_i + Z\tilde{n}_I \; . \tag{19}$$

Thus, for $\tilde{n}_i \sim \tilde{n}_e$ we have

$$\frac{\tilde{n}_I}{n_I} \sim \frac{Z}{Z_e}\frac{\tilde{n}_i}{n_i} \tag{20}$$

where

$$Z_e = n_I Z^2/n_i \ . \tag{21}$$

This implies that if we assume $Z > 1$ and $Z_e \sim 1$ the relative density fluctuations of impurities are larger than those of electrons and main ions.

For the electrons the perturbed momentum balance equation along the magnetic field,

$$0 = -\nabla_\parallel(\tilde{n}_e T_e) + e n_e \nabla_\parallel \tilde{\phi} \ ,$$

reduces to

$$\tilde{n}_e/n_e = e\tilde{\phi}/T_e \ . \tag{22}$$

The main ion and impurity ion mass conservation equations in the guiding center approximation are

$$\frac{\partial}{\partial t}\tilde{n}_j - c\frac{\nabla\tilde{\phi}\times B}{B^2}\cdot\nabla n_j + n_j\nabla_\parallel \tilde{u}_{j\parallel} = 0 \ , \tag{23}$$

where $j = i, I$. We define

$$\omega_{*I} \equiv k_y \frac{cT_i}{eB}\frac{d}{dx}\ln n_I \equiv \sigma_I \omega_{*i} \ , \qquad \sigma_I \equiv \frac{d\ln n_I/dx}{d\ln n_i/dx} \ , \tag{24}$$

and at first assume

$$\sigma_I \sim 1 \ . \tag{25}$$

A derivation of the full dispersion relation for arbitrary σ_I will be given in the appendix. Then, from Eq. (23) we have, recalling Eq. (20),

$$\frac{\tilde{n}_I}{n_I} = -\sigma_I \frac{\omega_{*i}}{\omega}\frac{e\tilde{\phi}}{T_i} + \frac{k_\parallel \tilde{u}_{I\parallel}}{\omega} \ , \tag{26}$$

and

$$0 = -\frac{\omega_{*i}}{\omega}\frac{e\tilde{\phi}}{T_i} + \frac{k_\parallel \tilde{u}_{i\parallel}}{\omega} \ . \tag{27}$$

In view of Eq. (18) it is realistic to consider the quantity

$$\frac{\omega}{\nu_{iI}}\frac{n_I m_I}{n_i m_i} \lesssim \frac{m_I}{m_i Z^2}(k_\parallel \lambda_i)^2 \tag{28}$$

as well below unity for $Z > 1$. In this limit the thermal electric force and the friction force resulting from collisions between the two ion populations play an important role in the ion momentum balance equations. From the impurity ion energy balance equation it can be seen that in the relevant limit we have $\tilde{T}_I = \tilde{T}_i$, which results from the strong collisional coupling of the impurity to the main ion population. On the other hand, the collisional transfer of thermal energy to the main ions may be neglected in view of the smallness of n_I/n_i. The total ion and impurity momentum conservation equation along the magnetic field gives

$$0 = -\nabla(\tilde{n}_i T_i + n_i \tilde{T}_i + e n_e \tilde{\phi})$$

and then

$$\frac{\tilde{n}_i}{n_i} = -\frac{e\tilde{\phi}}{T_i} - \frac{\tilde{T}_i}{T_i} \ . \tag{29}$$

This implies, for consistency with Eq. (27), that $\omega < |\omega_{*i}|$.

The longitudinal momentum balance equation for the impurity population is

$$0 = \alpha_{iI} n_i \nabla_\parallel \tilde{T}_i + \beta_{iI}\nu_{iI} m_i n_i (\tilde{u}_{i\parallel} - \tilde{u}_{I\parallel}) \ ,$$

so that

$$(\tilde{u}_{i\parallel} - \tilde{u}_{I\parallel}) = -i\frac{\alpha_{iI}}{\beta_{iI}}\frac{k_\parallel T_i}{\nu_{iI} m_i}\frac{\tilde{T}_i}{T_i} \ . \tag{30}$$

The α_{iI} term corresponds to the thermal force resulting from collisions between the main ion and impurity populations, and the β_{iI} term corresponds to the analogous friction force. The coefficients α_{iI} and β_{iI} are tabulated in Ref. 8 for a number of values of the impurity strength Z_e. In particular, $\beta_{iI} \sim 1$ for $Z_e \lesssim 1$ while α_{iI} is proportional to Z_e for small values of Z_e.

In order to obtain $\tilde{T}_i/T_i$ we refer to the ion thermal energy balance equation

$$\left[\frac{3}{2}\frac{\partial}{\partial t} - \left(\frac{\chi_i T_i}{\nu_i m_i}\right)\nabla_\parallel^2\right]\tilde{T}_i$$

$$= \tfrac{3}{2}c\frac{\nabla\tilde{\phi}\times B}{B^2}\cdot\nabla T_i - T_i\nabla\cdot\tilde{u}_{i\parallel} - \alpha_{iI}T_i\nabla_\parallel\cdot(\tilde{u}_{i\parallel} - \tilde{u}_{I\parallel}), \tag{31}$$

where $\chi_i \sim 1$ is tabulated in Ref. 8. Then, recalling Eqs. (27) and (30), we have

$$\left(\frac{3}{2} + i\hat{\chi}_i\frac{k_\parallel^2 T_i}{m_i\nu_i\omega}\right)\frac{\tilde{T}_i}{T_i} = \frac{e\tilde{\phi}}{T_i}\frac{\omega_{*i}}{\omega}(1 - \tfrac{3}{2}\eta_i) \ , \tag{32}$$

where $\hat{\chi}_i = \chi_i + \alpha_{iI}^2\nu_i/(\beta_{iI}\nu_{iI})$, and

$$\eta_i \equiv \frac{d\ln T_i/dx}{d\ln n_i/dx} \ . \tag{33}$$

We notice that the finite ion thermal conductivity and thermal force terms make $\tilde{T}_i$ out of phase with $\tilde{\phi}$ and can be expected to be responsible for the instability process of interest. Then, for simplicity we may consider the limit

$$k_\parallel^2 T_i/\omega\nu_i m_i > 1 \tag{34}$$

and obtain

$$\frac{\tilde{T}_i}{T_i} = -i\frac{\nu_i\omega_{*i}m_i}{\hat{\chi}_i k_\parallel^2 T_i}(1 - \tfrac{3}{2}\eta_i)\frac{e\tilde{\phi}}{T_i} \ . \tag{35}$$

A substitution of Eq. (35) into Eq. (30) gives

$$\frac{k_\parallel}{\omega}(\tilde{u}_{I\parallel} - \tilde{u}_{i\parallel}) = \hat{\alpha}_I\frac{\omega_{*i}}{\omega}(1 - \tfrac{3}{2}\eta_i)\frac{e\tilde{\phi}}{T_i} \ , \tag{36}$$

where $\hat{\alpha}_I = (\alpha_{iI}\nu_i)/(\beta_{iI}\nu_{iI}\hat{\chi}_i)$. The main ion density perturbation is obtained from Eqs. (29) and (35).

$$\frac{\tilde{n}_i}{n_i} = -\frac{e\tilde{\phi}}{T_i}\left(1 - i\frac{\nu_i\omega_{*i}m_i}{\hat{\chi}_i k_\parallel^2 T_i}(1 - \tfrac{3}{2}\eta_i)\right) \ , \tag{37}$$

and the impurity ion density perturbation follows from Eqs. (26), (27), and (36)

$$\frac{\tilde{n}_I}{n_I} = \frac{e\tilde{\phi}}{T_i}\frac{\omega_{*i}}{\omega}[(1 - \sigma_I) + \hat{\alpha}_I(1 - \tfrac{3}{2}\eta_i)] \ . \tag{38}$$

Consequently, the relevant dispersion relation is

$$1 + \frac{T_i}{T_e} + \frac{Z_e}{Z}\frac{\omega_{*i}}{\omega}[(\sigma_I - 1) + \hat{\alpha}_I(\tfrac{3}{2}\eta_i - 1)] = -i\frac{\nu_i\omega_{*i}m_i}{\hat{\chi}_i k_\parallel^2 T_i}(\tfrac{3}{2}\eta_i - 1) \ . \tag{39}$$

From this it is easy to evaluate the real and imaginary parts of the frequency and to see that the instability condition is

$$(\tfrac{3}{2}\eta_i - 1)[(\sigma_I - 1) + \hat{\alpha}_I(\tfrac{3}{2}\eta_i - 1)] > 0 \ . \tag{40}$$

The relevant growth rate $\gamma = \mathrm{Im}\,\omega$ is significant only if it is larger than the collisional rate of evolution of the equilibrium state that is discussed in Sec. VII. Then, we may estimate

$$\gamma \sim \frac{Z_e}{Z}\omega_{*i}^2 \frac{\nu_i}{k_\parallel^2 v_i^2} \ , \tag{41}$$

and require

$$\frac{k_y^2}{k_\parallel^2} > \left(\frac{Z}{Z_e}\right)^2 \frac{\nu_{iI}}{\nu_i} \tag{42}$$

a condition that is easily satisfied.

The inequality (40) indicates that if $\eta_i > \frac{2}{3}$, instability can occur for $\sigma_I > 0$, that is, starting from a state where impurities are accumulated at the center of the plasma column, and in particular for

$$\sigma_I > 1 - \hat{\alpha}_I(3\eta_i/2 - 1) \ . \tag{43}$$

Next, we reconsider the same problem in the limit where

$$\sigma_I \sim Z/Z_e > 1 \tag{44}$$

that is relevant to the discussion given in Sec. VII. In this case

$$\omega \sim \omega_{*i}$$

and Eq. (27) changes to

$$\frac{k_\parallel \tilde{u}_{i\parallel}}{\omega} = \frac{\tilde{n}_i}{n_i} + \frac{\omega_{*i}}{\omega}\frac{e\tilde{\phi}}{T_i} \tag{45}$$

and Eq. (35) to

$$\frac{\tilde{T}_i}{T_i} = i\frac{\nu_i m_i \omega_{*i}}{\hat{\chi}_i k_\parallel^2 T_i}\left(\frac{3}{2}\eta_i - 1 + \frac{\omega}{\omega_{*i}}\right)\frac{e\tilde{\phi}}{T_i} \ . \tag{46}$$

The main ion density perturbation is obtained from Eqs. (29) and (45)

$$\frac{\tilde{n}_i}{n_i} = -\frac{e\tilde{\phi}}{T_i}\left[1 + i\frac{\nu_i \omega_{*i} m_i}{k_\parallel^2 T_i \hat{\chi}_i}\left(\frac{\omega}{\omega_{*i}} + \tfrac{3}{2}\eta_i - 1\right)\right] \ . \tag{47}$$

In addition, Eq. (26) gives

$$\frac{\tilde{n}_I}{n_I} = -\frac{e\tilde{\phi}}{T_i}\frac{\omega_{*i}}{\omega}\sigma_I \ , \tag{48}$$

so that the relevant dispersion relation is

$$1 + \frac{T_i}{T_e} + \frac{Z_e}{Z}\sigma_I\frac{\omega_{*i}}{\omega} = -i\frac{\nu_i\omega_{*i}m_i}{k_\parallel^2 T_i\hat{\chi}_i}\left(\frac{\omega}{\omega_{*i}} + \tfrac{3}{2}\eta_i - 1\right) \tag{49}$$

and the instability condition becomes

$$\sigma_I\left[\frac{3}{2}\eta_i - 1 - \frac{Z_e}{Z}\sigma_I\left(1 + \frac{T_i}{T_e}\right)^{-1}\right] > 0 \ . \tag{50}$$

Then again, if $\eta_i > \frac{2}{3}$, as is realistic to expect, and for

$$0 < \sigma_I < \frac{Z}{Z_e}\left(1 + \frac{T_i}{T_e}\right)(\tfrac{3}{2}\eta_i - 1) \ , \tag{51}$$

we may argue that the relevant instability will tend to redistribute the impurity density, starting from a state of accumulation around the center of the plasma column, toward the outer edge of this as indicated by the following discussion.

We consider the quasi-linear particle and thermal energy transport across the magnetic field and notice that, since $\tilde{n}_e$ and $\tilde{\phi}$ are in phase, no net transport of electrons is produced. However, on the basis of the same theory, we expect that a rearrangement of the impurity spatial distribution, as well as transport of ion thermal energy, does result from the relevant modes. In order to estimate the flux, we consider the quasi-linear mass conservation equation for the main ions

$$\frac{\partial}{\partial t}n_i + \frac{\partial}{\partial x}\left\langle\left\langle i\frac{ck_y}{B}\tilde{n}_i\tilde{\phi}^* + \text{c.c.}\right\rangle\right\rangle = 0 \ , \tag{52}$$

where the brackets indicate a phase average, and notice that, from charge neutrality,

$$\frac{\partial}{\partial t}\ln n_I = -\frac{Z}{Z_e}\frac{\partial}{\partial t}\ln n_i \ . \tag{53}$$

This indicates that the rate of redistribution of the impurity population is considerably faster than that of the main ions for $Z/Z_e > 1$ and we can evaluate the former from the latter. In particular if

$$\frac{\partial n_i}{\partial t} = \frac{\partial}{\partial x}\left(D_i\frac{\partial}{\partial x}n_i\right) \ , \tag{54}$$

we also have

$$D_I = -(Z/Z_e\,\sigma_I)D_i \ , \tag{55}$$

so that the radial fluxes of main ions and impurity ions are in opposite directions. Then, if we consider the case where $\sigma_I \sim Z/Z_e$ and the limit (44), corresponding to Eq. (47)

$$\langle\langle i\tilde{n}_i\tilde{\phi}^* + \text{c.c.}\rangle\rangle = 2e|\tilde{\phi}|^2\frac{n_i}{T_i}\left(\hat{\chi}_i\frac{k_\parallel^2 T_i}{\omega_{*i}\nu_i m_i}\right)^{-1}\left(\frac{\omega_0}{\omega_{*i}} + \tfrac{3}{2}\eta_i - 1\right) \ , \tag{56}$$

where $\omega_0 = \mathrm{Re}\,\omega$. Combining Eqs. (52) and (56) we obtain

$$D_i = -\frac{cT_i}{eB}\left(\frac{\nu_i}{\hat{\chi}_i k_\parallel v_i}\right)\frac{2k_y^2\rho_i}{k_\parallel}\left|\frac{e\tilde{\phi}}{T_i}\right|^2\left(\frac{\omega_0}{\omega_{*i}} + \tfrac{3}{2}\eta_i - 1\right) \ , \tag{57}$$

so that, recalling Eq. (49),

$$D_i \propto -\left[\frac{3}{2}\eta_i - 1 - \frac{Z_e}{Z}\left(1 + \frac{T_i}{T_e}\right)^{-1}\sigma_I\right] \ . \tag{58}$$

We see that if the condition for instability (51) is satisfied we have $D_I > 0$ and the flow of impurities is directed outward. Since $\omega/\omega_{*i} \sim \sigma_I Z_e/Z$, we may argue that, in view of Eq. (57), if we start from an equilibrium state where impurities are accumulated around the center of the plasma column, their outflow will continue for $\eta_i > \frac{2}{3}$ until $\sigma_I \sim 1 - \hat{\alpha}_I(3\eta_i/2 - 1)$ as indicated by Eq. (43). The ion thermal energy transport can be evaluated from the contributions of the $\mathbf{E} \times \mathbf{B}$ drift in the quasi-linear thermal energy balance equation for the main ions

$$\frac{\partial}{\partial t}T_i + \frac{\partial}{\partial x}\left\langle\left\langle i\frac{ck_y}{B}\tilde{T}_i\tilde{\phi}^* + \text{c.c.}\right\rangle\right\rangle = 0 \ . \tag{59}$$

Then, referring to Eq. (29), we have

$$\langle\langle i\tilde{T}_i\tilde{\phi}^* + \text{c.c.}\rangle\rangle = \frac{T_i}{n_i}\langle\langle -i\tilde{n}_i\tilde{\phi}^* + \text{c.c.}\rangle\rangle \ , \tag{60}$$

so that

$$\frac{\partial}{\partial t}T_i=\frac{\partial}{\partial x}\left(\frac{D_i}{\eta_i}\frac{\partial}{\partial x}T_i\right) . \tag{61}$$

The quasi-linear diffusion coefficient is significant only when it is larger than the collisional one. The latter can be derived from Sec. VII.

$$D_{cI}\approx\frac{Z}{Z_e\sigma_I}\frac{\nu_{iI}T_i}{\Omega_i^2 m_i} \tag{62}$$

so that we obtain

$$\frac{D_I}{D_{cI}}\approx\frac{2}{\hat{\chi}_i}\frac{k_y^2}{k_\parallel^2}\frac{\nu_i}{\nu_{iI}}\left|\frac{e\tilde{\phi}}{T_i}\right|^2 . \tag{63}$$

Thus, referring to Eq. (42), it is possible to have $D_I > D_{cI}$ for relatively low fluctuation levels $|e\tilde{\phi}/T_i|$. As for the possible values of $|e\tilde{\phi}/T_i|$ it is difficult to indicate, in general, those that correspond to the saturation level of the relevant instability in a given experimental situation. In order to obtain an estimate for the fluctuation amplitude that may be reached, we may assume that the impurity density distribution is changed by the instability until the linear and the nonlinear terms within the mass conservation equation become of the same order

$$\tilde{u}_x\frac{d}{dx}n_I\sim\tilde{\mathbf{u}}\cdot\nabla\tilde{n}_I ,$$

where $u=\mathbf{B}\times\nabla\tilde{\phi}/B^2$, so that

$$\tilde{n}_I/n_I\sim\lambda_x/r_{nI} , \tag{64}$$

where λ_x is the typical radial wavelength and r_{nI} is the impurity density scale length. Then, referring to Eqs. (21) and (29), we have $(e\tilde{\phi}/T_i)\sim(\tilde{n}_i/n_i)\sim(Z_e/Z)(\tilde{n}_I/n_I)$ so that

$$\frac{e\tilde{\phi}}{T_i}\sim\frac{Z_e}{Z}\frac{\lambda_x}{r_{nI}} . \tag{65}$$

Notice that D_I can be rewritten as

$$D_I\approx\frac{Z}{Z_e\sigma_I}\frac{2\gamma}{\omega_0}k_y r_{ni}\frac{cT_i}{eB}\left|\frac{e\tilde{\phi}}{T_i}\right|^2 , \tag{66}$$

where r_{ni} is the main ion density scale length. Combining Eqs. (65) and (66) we obtain

$$D_I\approx 2\gamma\lambda_x^2 . \tag{67}$$

Now, for the sake of completeness, we consider the limit where[2,6]

$$\frac{\nu_{iI}}{\omega}<\frac{m_I n_I}{m_i n_i},\ \frac{k_\parallel^2 T_i}{\omega^2 m_i} \tag{68}$$

and obtain the closest collisional version to the impurity drift instability which we shall treat in Sec. IV. These limits imply that $m_I > Z^2 m_i$ and that $Z_e < 1$. Thus, Eq. (68) is appropriate to the case of a plasma with a small amount of low-Z impurities. Then, the longitudinal momentum balance equation for the main ions is

$$0=\nabla_\parallel(\tilde{n}_i T_i+n_i\tilde{T}_i)-en_i\nabla_\parallel\tilde{\phi} \tag{69}$$

so that

$$\frac{\tilde{n}_i}{n_i}+\frac{e\tilde{\phi}}{T_i}+\frac{\tilde{T}_i}{T_i}=0 . \tag{70}$$

Substituting Eqs. (22), (26), and (70), neglecting $\tilde{u}_{I\parallel}$, into the quasi-neutrality condition (19) we obtain

$$1+\frac{T_i}{T_e}+\frac{Z_e}{Z}\sigma_I\frac{\omega_{*i}}{\omega}=\frac{\tilde{T}_i}{e\tilde{\phi}} , \tag{71}$$

where from the ion energy balance equation (31) in the limit (34),

$$\frac{\tilde{T}_i}{e\tilde{\phi}}=i\frac{\nu_i m_i\omega_{*i}}{\chi_i k_\parallel^2 T_i}\left((\tfrac{3}{2}\eta_i-1)+\frac{\omega}{\omega_{*i}}\right) . \tag{72}$$

From this it is easy to evaluate the real and imaginary parts of the frequency and to see that the instability condition is, assuming $dn_i/dx<0$,

$$\frac{-1}{\hat{\omega}_0}\frac{Z}{Z_e\sigma_I}(\tfrac{3}{2}\eta_i-1)\left(1+\frac{T_i}{T_e}\right)>1 , \tag{73}$$

where

$$\hat{\omega}_0=\omega\left(\sigma_I\frac{Z_e}{Z}\omega_{*i}\right)^{-1}\left(1+\frac{T_i}{T_e}\right) .$$

As indicated by Eq. (73), the relevant instability is excited when impurities are accumulated at the center of the plasma column if $\eta_i>\frac{2}{3}$ and vice-versa. The discussion of the quasi-linear transport due to this mode can easily be inferred from the one given in the next section.

If the high temperature plasma being considered is surrounded by a cold plasma blanket, the cold ions will have some similarities in behavior to the impurity ions considered here, in that they constitute a distinct population with thermal velocity smaller than that of the main ions. In particular, we might expect analogous modes to develop, with the possibility that these have the effect of transporting the cold ions inward into the hot plasma region if the relevant temperature gradient, represented by η_i, is sufficiently flat.

IV. RESONANT SHORT-WAVELENGTH MODES

A. Linear analysis

Now we consider the case where[4,6]

$$\lambda_\parallel\ll\lambda_i , \tag{74}$$

and

$$\lambda_\parallel\ll L ,$$

so that we may ignore the effects of the magnetic field periodicity and refer to a one-dimensional, plane equilibrium, and we ignore the influence of collisions on the modes that can be excited. The relevant frequency range is given by (17). The equilibrium distribution function is taken to be of the form

$$f_j^0=f_{Mj}(x)(1+\hat{f}_j) , \tag{75}$$

where, for kinetic energy ϵ,

$$f_{Mj}(x)=\frac{n_j(x)}{[2\pi T_j(x)/m_j]^{3/2}}\exp\left(\frac{-\epsilon}{T_j(x)}\right) ,$$

and

$$\hat{f}_j=\frac{v_y}{\Omega_j}\left[\frac{d\ln n_j}{dx}-\frac{d\ln T_j}{dx}\left(\frac{3}{2}-\frac{\epsilon}{T_j}\right)\right] .$$

The perturbed densities in the frequency range being considered are obtained from integration of the perturbed Vlasov equation along particle orbits and, for the frequency range (17), can be written as[9]

$$\tilde{n}_e = n_e \frac{e\tilde{\phi}}{T_e}, \qquad (76)$$

$$\tilde{n}_i = -n_i \frac{e\tilde{\phi}}{T_i}\left\{1 - \left[\left(1 - \frac{\omega_{*i}}{\omega}\{1-\eta_i[\tfrac{1}{2}+F(b_i)]\}\right)(1+W_i) - \frac{1}{2}\frac{\omega_{*i}}{\omega}\eta_i \frac{\omega^2 m_i}{k_\parallel^2 T_i} W_i\right] I_0(b_i)\exp(-b_i)\right\}, \qquad (77)$$

and

$$\tilde{n}_I = n_I \frac{e\tilde{\phi}}{T_i}\left\{\frac{ZT_i}{T_I}W_I - \frac{\omega_{*I}}{\omega}\left[1 + W_I - \tfrac{1}{2}\eta_I\left(1 + W_I - \frac{\omega^2 m_I}{k_\parallel^2 T_I}W_I\right)\right]\right\}, \qquad (78)$$

where $b_j \equiv (1/2)k_y^2\rho_j^2$, $F(b_i) \equiv b_i - b_i I_1(b_i)/I_0(b_i)$, I_0 and I_1 are modified Bessel functions of the first kind, $\omega_{*i} \equiv k_y(cT_i/eB)\times(d\ln n_i/dx)$, $\omega_{Tj} \equiv \eta_j\omega_{*j}$, $\eta_j \equiv (d\ln T_j/dx)/(d\ln n_j/dx)$, $\omega_{*I} = \sigma_I\omega_{*i}$, $\sigma_I \equiv (d\ln n_I/dx)/(d\ln n_i/dx)$, and

$$W_j = W\left(\frac{\omega}{k_\parallel v_j}\right) \equiv -\pi^{-1/2}\int_{-\infty}^{\infty} d\xi\, \xi e^{-\xi^2} \Big/ \left(\xi - \frac{\omega}{k_\parallel v_j}\right). \qquad (79)$$

We have neglected finite gyroradius corrections for the impurity population as we consider

$$b_I \sim b_i(m_I/m_i Z^2) < 1. \qquad (80)$$

For

$$v_I \lesssim \omega/k_\parallel < v_i \qquad (81)$$

we have

$$W_i \simeq -1 + 2\left(\frac{\omega}{k_\parallel v_i}\right)^2 - i\pi^{1/2}\left(\frac{\omega}{|k_\parallel| v_i}\right),$$

and

$$\tilde{n}_i = -\frac{e\tilde{\phi}}{T_i}n_i\left[1 + i\pi^{1/2}\frac{\omega}{|k_\parallel| v_i}I_0(b_i) \times \exp(-b_i)\left(1 - \frac{\omega_{*i}}{\omega}\{1-\eta_i[\tfrac{1}{2}+F(b_i)]\}\right)\right]. \qquad (82)$$

Then, the quasi-neutrality condition

$$\tilde{n}_e = \tilde{n}_i + Z\tilde{n}_I$$

may be written as

$$-\frac{Z_e}{Z}\frac{\omega_{*I}}{\omega}\left\{1 + W_I - \tfrac{1}{2}\eta_I\left[1 - \left(\frac{\omega^2 m_I}{k_\parallel^2 T_I}W_I\right) + W_I\right]\right\}$$
$$= \left(1 + \frac{n_e T_i}{n_i T_e} - Z_e\frac{T_i}{T_I}W_I\right) + I_0(b_i)\exp(-b_i) i\pi^{1/2}\frac{\omega}{|k_\parallel| v_i} \times\left(1 - \frac{\omega_{*i}}{\omega}\{1-\eta_i[\tfrac{1}{2}+F(b_i)]\}\right), \qquad (83)$$

where Z_e is given by Eq. (21).

In deriving the result (83), we have neglected magnetic curvature drift effects. To include this effect for the impurity ions, we define the impurity ion curvature drift frequency $\bar{\omega}_{DI} \equiv k_y g_I m_I c/(eB_0)$, where g_I is an effective gravitational field, acting on the impurity ions, that is introduced in order to simulate the drift of the impurity ions in an unfavorable magnetic curvature region. We note that $\bar{\omega}_{DI} = \omega_{*I}(r_{nI}/R_0)\alpha_D$, where α_D is a dimensionless constant of order unity, $r_{nj} \equiv -(d\ln n_j/dx)^{-1}$, and R_0 is the radius of magnetic curvature which for a toroidal configuration (see Sec. V) is $L/(2\pi q)$, q being the inverse rationalized rotational transform, and $g_I \simeq -\alpha_D T_i/(m_I R_0)$. Including this term in the analysis, the left-hand side of Eq. (83) becomes, for $\epsilon_0 = r/R_0$,

$$-\frac{Z_e}{Z}\frac{\omega_{*I}}{\omega}\left\{1 + W_I - \tfrac{1}{2}\eta_I\left[1 - \left(\frac{\omega^2 m_I}{k_\parallel^2 T_I}W_I\right) + W_I\right] + \left(\frac{r_{nI}}{r_0}\right)\epsilon_0\alpha_D\left(1 - \frac{\omega_{*I}}{\omega}\frac{T_I}{ZT_i}(1+\eta_I)\right)\right\},$$

and we see that this would add only a correction term of higher order in ϵ_0 to the value of ω that can be obtained from (83). The magnetic curvature effect for the main ions is even less important, so that we are justified in neglecting them for both ion species.

At first we consider the case where $\eta_I \lesssim 1$ and $\omega/(k_\parallel v_I) > 1$ so that W_I can be taken as real. Then, marginal stability corresponds to

$$\omega = \omega_{*i}\{1 - \eta_i[\tfrac{1}{2} + F(b_i)]\}. \qquad (84)$$

We may distinguish an "impurity-sound mode" corresponding to relatively large values of Z_e, such that

$$Z_e\frac{T_i}{T_I}W_I \sim 1 > \frac{Z_e}{Z}\frac{\omega_{*I}}{\omega}$$

and, since $W_I \lesssim 0.2$,

$$Z_e \gtrsim 5(T_I/T_i), \qquad (85)$$

where

$$\omega > (5/Z)(T_I/T_i)\omega_{*I}. \qquad (86)$$

When these conditions are met, the real part of the frequency is given by

$$1 + \frac{n_e}{n_i}\frac{T_i}{T_e} - Z_e\frac{T_i}{T_I}W_I = 0, \qquad (87)$$

and the imaginary part by

$$\left(-Z_e\frac{T_i}{T_I}\frac{dW_I}{d\omega}\omega\right)\frac{\delta\omega}{\omega} + i\pi^{1/2}\frac{\omega}{|k_\parallel| v_i}I_0(b_i)\exp(-b_i) \times\left(1 - \frac{\omega_{*i}}{\omega}\{1-\eta_i[\tfrac{1}{2}+F(b_i)]\}\right), \qquad (88)$$

for $\delta\omega \simeq i\,\mathrm{Im}\,\omega$. It is easy to verify that the coefficient of $\delta\omega/\omega$ is positive and finite. Thus, the instability condition is

$$\frac{\omega_{*i}}{\omega}\{1 - \eta_i[\tfrac{1}{2}+F(b_i)]\} > 1, \qquad (89)$$

and we see that σ_I does not affect it directly, as $(5T_I/T_i)\omega_{*I}/Z < \omega < \omega_{*i}$, while $1 - \eta_i[(1/2) + F(b_i)]$ can have either sign.

For lower values of Z_e the "impurity drift mode" discussed in the previous section is the only one possible and the relevant instability condition is

$$\frac{Z}{Z_e\sigma_I}\left(1 - \frac{\eta_i}{\eta_{ic}}\right) + \left(1 + \frac{n_e}{n_i}\frac{T_i}{T_e} - Z_e\frac{T_i}{T_I}\omega_I\right)^{-1} < 0, \qquad (90)$$

where

$$\eta_{ic} = 2/[1 + 2F(b_i)]. \qquad (91)$$

Notice that $F(b_i) \simeq b_i$ for $b_i < 1$ and $F(b_i) \simeq \frac{1}{2}$ for $b_i > 1$, so that the condition $\eta > \eta_{ic}$ is more easily satisfied for

$b_i > 1$ than for $b_i < 1$.

This instability is reminiscent of the ion drift mode that is driven by a sufficiently large temperature gradient since it depends on ion Landau damping and is also driven by the gradient of the mean longitudinal velocity that, in the present case, results from a gradient of the mean ion mass number. Condition (90) indicates that the relevant mode can be excited for σ_I positive or negative depending on the value of η_i. For the impurity drift mode to occur for a main ion temperature gradient sufficiently small so that $\eta_i < \eta_{ic}$, the density gradient of impurity ions must be in the opposite sense to that of the main ions and not too large. This could be the case if the impurity ions were concentrated near the plasma boundary. On the other hand, for a sufficiently large main ion temperature gradient, such that $\eta_i > \eta_{ic}$, the instability could occur for an impurity ion density gradient in the same sense as that of the main ions, which would be the case if the impurity ions were concentrated at the plasma center. In these cases $|\omega| < |\omega_{*i}|$ and the worst growth, obtained by taking the limiting value of $k_\parallel$ in Eq. (83), is

$$\mathrm{Re}\,\omega \sim \mathrm{Im}\,\omega \sim Z\omega_{*i}\frac{d\ln n_I/dx}{d\ln n_i/dx}\,. \qquad (92)$$

Now, we consider the limit

$$\eta_I \sim \omega^2 m_I/k_\parallel^2 T_I > 1$$

that is relevant to our discussion of the evolution of the impurity density profile. In this case Eq. (83) reduces to

$$1+\frac{n_e T_i}{n_i T_e}+\frac{Z_e}{Z}\sigma_I\frac{\omega_{*i}}{\omega}\left(1+\eta_I\frac{k_\parallel^2 T_I}{\omega^2 m_I}\right)=0 \qquad (93)$$

and a fluid-like (nonresonant) instability can be found. Equation (93) has an unstable root if

$$\left[1+\tfrac{27}{2}\,\eta_I\frac{k_\parallel^2 T_i}{\omega_{*i}^2 m_I}\left(\frac{Z}{Z_e\sigma_I}\right)^2\left(1+\frac{n_e T_i}{n_i T_e}\right)^2\right]^2>1\,, \qquad (94)$$

and we may argue that, as a result, a transport of impurities toward the outer region can occur until the impurity density gradient has changed sign (meaning $\eta_I \equiv \omega_{*I}/\omega_{TI} < 0$) and a condition for marginal stability is reached.

Now, we notice that the plasma may not sustain relatively large impurity density gradients, such as those predicted by the collisional transport theory, for which $\sigma_I \sim Z$. In fact, if

$$\sigma_I \simeq \frac{Z}{Z_e}[\eta_i F(b_i)-1]I_0(b_i)\exp(-b_i)\,, \qquad (95)$$

the instability associated with Eq. (94) is of the fluid type in the sense that it does not involve a wave-particle resonance and occurs in the limit

$$v_I < v_i < \omega/k_\parallel < v_e\,,$$

for which the dispersion relation is

$$0=\frac{T_i n_e}{T_e n_i}+[1-I_0(b_i)\exp(-b_i)]+\frac{\omega_{*i}}{\omega}\left(\sigma_I\frac{Z_e}{Z}+I_0(b_i)\exp(-b_i)[1-\eta_i F(b_i)]\right)-\frac{1}{2}\left(\frac{k_\parallel v_i}{\omega}\right)^2\left(I_0(b_i)\exp(-b_i)+Z_e\frac{m_i}{m_I}\right.$$
$$\left.-\frac{\omega_{*i}}{\omega}[I_0(b_i)\exp(-b_i)]\{1+\eta_i[1-F(b_i)]\}+\sigma_I\frac{Z_e}{Z}\frac{T_I}{T_i}\frac{m_i}{m_I}\right)\,. \qquad (96)$$

When the condition (95) is satisfied, the term proportional to $1/\omega$ in (96) is zero, and the resulting cubic equation has an unstable root if

$$\left(\frac{\omega_{*i}}{k_\parallel v_i}\right)^2>\frac{2}{27}[I_0(b_i)\exp(-b_i)]^3\left(\frac{T_i n_e}{T_e n_i}+1-I_0(b_i)\exp(-b_i)\right)^{-1}[I_0(b_i)\exp(-b_i)]\left(1+\eta_i[1-F(b_i)]+\sigma_I\frac{Z_e}{Z}\frac{m_i}{m_I}\frac{T_I}{T_i}\right)^{-2} \qquad (97)$$

a condition which is satisfied whenever $\omega > k_\parallel v_i$. If we take the limit of this condition and omit the term in (96) proportional to $1/\omega^2$ entirely, we may solve for the frequency ω, to obtain

$$\omega^3=-\tfrac{1}{2}k_\parallel^2 v_i^2\omega_{*i}\left\{I_0(b_i)\exp(-b_i)\left[1+\eta_i\left(1-b_i+b_i\frac{I_1(b_i)}{I_0(b_i)}\right)\right]+\sigma_I\frac{Z_e}{Z}\frac{m_i}{m_I}\frac{T_I}{T_i}\right\}\left(\frac{T_i n_e}{T_e n_i}+1-I_0(b_i)\exp(-b_i)\right)^{-1}.$$

Condition (95) requires that σ_I be of order $|\sigma_I| \sim Z/Z_e$ if $b_i \lesssim 1$ and $\eta_i \lesssim 1$. On the other hand, if $b_i \gg 1$, the condition (95) becomes $\sigma_I \simeq (Z/Z_e)(\eta_i/2-1)(2\pi b_i)^{-1/2}$, which can be satisfied for $|\sigma_I| < Z/Z_e$ although in this case, finite b_I corrections should be included. We notice that the signs of σ_I required to satisfy the condition (95) for $b_i \lesssim 1$ are opposite to those which tend to be produced by collisions (see Sec. VII) for large and small values of η_i.

B. Quasi-linear effects

We consider the quasi-linear particle and thermal energy transport due to the impurity modes that we have been considering. For simplicity, we neglect all finite gyroradius effects. Then, the relevant quasi-linear equation is

$$\frac{\partial}{\partial t}f_j^0=-2i\sum_{\mathbf{k}}\frac{c}{B}\frac{\partial}{\partial x}k_y\tilde{\phi}_{-\mathbf{k}}\tilde{f}_{j\mathbf{k}}+\frac{e}{m}k_\parallel\tilde{\phi}_{-\mathbf{k}}\frac{\partial}{\partial v_\parallel}\tilde{f}_{j\mathbf{k}}\,, \qquad (98)$$

where we have written $f_j = f_j^0+\tilde{f}_j$, with $\tilde{f}_j/f_j^0 \sim e_j\tilde{\phi}/T_j \ll 1$, and

$$\tilde{f}_{j\mathbf{k}}=\frac{-1}{\omega-k_\parallel v_\parallel}\left(\frac{e_j}{m_j}k_\parallel\tilde{\phi}_{\mathbf{k}}\frac{\partial}{\partial v_\parallel}f_j^0+\frac{c}{B}k_y\tilde{\phi}_{\mathbf{k}}\frac{\partial}{\partial x}f_j^0\right)\,. \qquad (99)$$

We may separate the denominator in (98) into a resonant and a nonresonant part

$$\frac{1}{\omega-k_\parallel v_\parallel}=-i\pi\delta(\omega-k_\parallel v_\parallel)+\frac{\omega_0-k_\parallel v_\parallel-i\gamma}{(\omega_0-k_\parallel v_\parallel)^2+\gamma^2}\,. \qquad (100)$$

Evidently, for instabilities which are unstable due to the mode-particle resonance term for the species whose diffusion is being calculated, the resonant diffusion term is dominant when the resonance condition $\omega = k_{\parallel}v_{\parallel}$ is met, and the nonresonant diffusion term otherwise. Integrating (98) over velocity space, we have

$$\frac{\partial}{\partial t}n_j^0 = -\frac{c}{B}\frac{\partial}{\partial x}\sum_{\mathbf{k}} 2ik_y\tilde{\phi}_{-\mathbf{k}}\tilde{n}_{j\mathbf{k}} \equiv -\frac{\partial}{\partial x}\Gamma_j \qquad (101)$$

and

$$\frac{\partial}{\partial t}W_{\perp j}^0 = -\frac{c}{B}\frac{\partial}{\partial x}\sum_{\mathbf{k}} 2ik_y\tilde{\phi}_{-\mathbf{k}}\tilde{W}_{\perp j\mathbf{k}} \equiv -\frac{\partial}{\partial x}q_{\perp j}\ , \qquad (102)$$

where $W_{\perp j}^0 = \int d^3\mathbf{v}\,\frac{1}{2}m_j v_\perp^2 f_j^0$, $\tilde{W}_{\perp j\mathbf{k}} = \int d^3\mathbf{v}\,\frac{1}{2}m_j v_\perp^2\tilde{f}_{j\mathbf{k}}$.

Now, we recall the quasi-neutrality condition (19) and derive

$$\Gamma_I = -(1/Z)\Gamma_i \qquad (103)$$

from the expression for Γ_i. Then, we obtain

$$\Gamma_i = \frac{2\pi^{1/2}}{r_{ni}}n_i D_{Bi}\sum_{\mathbf{k}}\left|\frac{e\tilde{\phi}_{\mathbf{k}}}{T_i}\right|^2\frac{k_y^2\rho_i}{2|k_\parallel|}\left(1-\frac{1}{2}\eta_i-\frac{\omega}{\omega_{*i}}\right), \qquad (104)$$

where $r_{ni} = -(d\ln n_i/dx)^{-1}$ and

$$D_{Bi} = cT_i/eB\ . \qquad (105)$$

Therefore, the impurity flux is outgoing if

$$\eta_i > 2(1-\omega/\omega_{*i})\ , \qquad (106)$$

and Γ_I will go to zero if

$$1-\frac{1}{2}\eta_i + \frac{Z_e}{Z}\sigma_I\left(1+\frac{T_i}{T_e}\right)^{-1} = 0\ , \qquad (107)$$

taking for ω the impurity drift mode root. In the case of the impurity sound mode, the instability condition requires

$$\frac{\omega_{*i}}{\omega}\left(1-\frac{\eta_i}{2}\right) > 1\ . \qquad (108)$$

Then, if $\eta_i > 2$, the relevant root of the dispersion relation corresponds to $\omega_{*i}/\omega < 0$ that implies outgoing impurity flux, on the basis of Eq. (104).

Next, we turn to the question of main ion thermal energy transport. The expression for $\tilde{W}_{\perp i}$ corresponding to (82) is

$$\tilde{W}_{\perp i} = -e\tilde{\phi}n_i\left[1+i\pi^{1/2}\frac{\omega}{|k_\parallel|v_i}\left(1-\frac{\omega_{*i}}{\omega}(1+\frac{1}{2}\eta_i)\right)\right]\ , \qquad (109)$$

and thus

$$q_{\perp i} \simeq \frac{2\pi^{1/2}}{r_{Ti}}(n_iT_i)D_{Bi}\sum_{\mathbf{k}}\left|\frac{e\tilde{\phi}_{\mathbf{k}}}{T_i}\right|^2\frac{k_y^2\rho_i}{2\eta_i|k_\parallel|}\left(1+\frac{\eta_i}{2}-\frac{\omega}{\omega_{*i}}\right)\ , \qquad (110)$$

where $r_{Ti} \equiv -(d\ln T_i/dx)^{-1}$. Then, we see that the ion heat transport is always outward. If we define an impurity "diffusion" coefficient

$$D_I \equiv \frac{\Gamma_I}{n_I}r_n = -\frac{Z}{Z_e}\left(\frac{\Gamma_i}{n_i}r_n\right)\ , \qquad (111)$$

where $r_{ni} \sim r_{nI} \equiv r_n$, and an ion thermal diffusion coefficient

$$D_{Ti} = (q_{\perp i}/n_iT_i)r_{Ti}\ , \qquad (112)$$

we can see that

$$D_I \sim (Z/Z_e)D_{Ti}\ . \qquad (113)$$

Therefore, the rate of change of n_I can be expected to be faster than that of either n_i or T_i and, as indicated in Sec. III, the evolution of σ_I will be such as to stabilize the various impurity modes, with η_i little affected on the relevant time scale.

We may summarize the evolution of the equilibrium on the basis of the results that have been presented so far as follows. For typical collisional equilibria (see Sec. VII), if it happens that the impurity ions become concentrated at the center of the plasma, with $\sigma_I \gtrsim 1$, the impurity drift mode can be unstable for $\eta_i \sim 1$, and serve to carry impurities outward (and main ions inward) until $\sigma_I \sim 0$ and this mode is stabilized. At this point the impurity (sound) mode that is driven by a temperature gradient of this population and is described by Eq. (93), will continue to carry impurities outward against the developing positive impurity density gradient, until it becomes stable for $\sigma_I \sim -1$. Thus, we may envision a stationary state in which

$$\eta_{ic} < \eta_i < \eta_c\ ,$$

η_c indicating the critical temperature gradient below which the collisional transport of impurities is inward (see Sec. VII), and the impurity temperature gradient driven instability is excited with such an amplitude as to produce an outward flow of impurities that exactly compensates the one due to collisions. At this point we will be left with the impurities concentrated at the outside of the plasma. We denote this process[1] as "impurity decontamination," and we expect that the value of η_i should not change appreciably during this process.

Finally, we may anticipate a comparison of these modes with the longer-parallel wavelength standing modes which will be investigated in Sec. V. The critical value η_{ic}, for which the sign of σ_I corresponding to instability of impurity drift modes changes, has values for η_{ic} approximately equal to 1 to 2 for the short-wavelength traveling modes and for η_{ic} approximately equal to $\frac{2}{3}$ to 1 for the standing modes. Requiring $\eta_i \geq \eta_{ic}$, the latter range is the more realistic of the two, and, in addition, the typical values of the transverse wavelengths that correspond to the lower values of η_{ic} are larger in the case of toroidal standing modes. Also, the basic impurity drift and impurity sound instabilities as found in this section for $\eta_I \lesssim 1$ are dependent on wave–particle resonances involving a small region of velocity space, whereas the corresponding instabilities in Sec. V can be nonresonant. We therefore expect the latter to have stronger effects.

V. COLLISIONLESS TRAPPED PARTICLE MODES

A. General dispersion relation

In this section we shall consider collisionless modes which can be excited in a two-dimensional confinement configuration such as an axisymmetric torus due to the presence of impurity ions.[1,3,10] The effects of collisions on the relevant modes will be considered in Sec. VI. When referring to an axisymmetric toroidal configuration, ζ and θ denote, respectively, the toroidal and poloidal angular coordinates, r is the minor radius for a given magnetic surface, and R_0 is the radius of the magnetic axis. The magnetic field is represented by $B=[B_0\mathbf{e}_\zeta+B_\theta^0(r)\mathbf{e}_\theta]/h(\theta)$, where $h(\theta)\equiv 1+\epsilon_0\cos\theta$, and we consider $\epsilon_0\equiv r/R_0\ll 1$. For simplicity, we have assumed that the magnetic surfaces are concentric and have circular cross section. The inverse rationalized rotational transform is then $q(r)=rB_\zeta/R_0B_\theta^0\simeq d\zeta/d\theta|_{\text{field line}}$, so that $L=2\pi qR_0$ and $\Delta B/B=2\epsilon_0$. As coordinates in velocity space we shall use either $v_\parallel$ and $v_\perp$, the velocity components parallel and perpendicular to the magnetic field, or the kinetic energy $\epsilon\equiv(\frac{1}{2})m_j(v_\parallel^2+v_\perp^2)$, the magnetic moment $\mu\equiv(\frac{1}{2})m_jv_\perp^2/B$, and the dimensionless pitch angle variable $\Lambda\equiv\mu B_0/\epsilon$. We assume that no electric field exists in the equilibrium and note that trapped particles correspond to $1-\epsilon_0\le\Lambda\le 1+\epsilon_0$, while circulating particles correspond to $0\le\Lambda<1-\epsilon_0$. We define the bounce period $\tau_b=2\int_{-\theta_0}^{\theta_0}d\theta/|\dot\theta|$ and the bounce frequency $\omega_b(\epsilon,\Lambda)\equiv 2\pi/\tau_b$ for trapped particles, where $\dot\theta\simeq\dot\zeta/q\simeq v_\parallel/(qR_0)$ and $\pm\theta_0=\pm\arccos[(\Lambda-1)/\epsilon_0]$ correspond to the orbit turning points. Likewise, we define the transit period $\tau_t=\int_{-\pi}^{\pi}d\theta/\dot\theta$ and the transit frequency $\omega_t(\epsilon,\Lambda)\equiv 2\pi/\tau_t$ for circulating particles. The relevant averages over a Maxwellian distribution are $\langle\omega_t\rangle_j\simeq\bar\omega_{tj}\equiv v_j/(qR_0)$ and $\langle\omega_b\rangle_j\simeq(\epsilon_0/2)^{1/2}\bar\omega_{tj}\ll\bar\omega_{tj}$.

For the types of modes we shall consider, the perturbed electrostatic potential may be represented as[10]

$$\tilde\Phi=\tilde\phi_{m^0,n^0}(\theta,S)\exp\{-i\omega t+in^0[\zeta-q(r)\theta]+iS(r)F(\theta)\} \tag{114}$$

in the neighborhood of a rational surface $r=r_0$, where $\tilde\phi_{m^0,n^0}(\theta,S)$ is periodic in θ with period 2π, while $F(\theta)$ is monotonic in θ so that $F(\theta)=\int_0^\theta d\theta'\,G(\theta')$, $G(\theta)$ also being periodic and even around $\theta=0$, with $F(\theta+2\pi)=F(\theta)+2\pi$. The surface $r=r_0$ is such that $q(r_0)=m^0/n^0\equiv q_0$, m^0 and n^0 being integers, and we have defined the radial variable $S(r)\equiv n^0[q(r)-q_0]$. Thus, $\tilde\phi_m(\theta)\equiv\tilde\phi_{m^0,n^0}(\theta,0)$ represents the mode amplitude modulation along a given magnetic field line for $r=r_0$.

The collisionless impurity modes that we shall consider are of the *odd* type, as $\tilde\phi_m(\theta)$ is odd in θ around $\theta=0$, and have frequencies such that

$$\langle\omega_t\rangle_I<\omega<\langle\omega_b\rangle_i\ll\langle\omega_b\rangle_e\ . \tag{115}$$

These modes are standing along the magnetic field lines and have $\tilde\phi_m(\theta=\pm\pi)=0$. In this case, we may choose $G(\theta)\simeq 2\pi\delta(\theta\pm\pi)$, so that $F(\theta)\simeq 0$ for trapped particles. In addition, $\tilde\phi_{m^0,n^0}(\theta,S)$ is taken, as a function of $r-r_0$, to be localized over a distance $\Delta r<r_0$, which we take to be related to the scale distances $r_{nj}\equiv-(d\ln n_j/dr)^{-1}$ and $r_{Tj}\equiv-(d\ln T_j/dr)^{-1}$. Since we consider relatively large values of m^0, such that the spacing between mode-rational surfaces $\Delta r_s\equiv(n^0dq/dr)^{-1}\ll r_{nj}\sim r_{Tj}$, $\tilde\phi_{m^0,n^0}(\theta,S)$ can be taken as a nearly periodic function in r with period $\Delta r_s<\Delta r$, or in S with period one, as can be verified *a posteriori*. Thus, we refer to the interval $-\frac{1}{2}\le S\le\frac{1}{2}$ and notice that for the modes of interest $(\Delta r_s)^2>\rho_{bi}^2$, ρ_{bi} being the average width of a trapped main ion banana orbit ($\rho_{bi}\sim\epsilon_0^{1/2}\rho_iB/B_\theta$).

The equilibrium particle distribution functions are assumed to be of the form $f_j^0=f_{Mj}(r)[1+\hat f_j]$, where

$$f_{Mj}(r)=\frac{n_j(r)}{[2\pi T_j(r)/m_j]^{3/2}}\exp[-\epsilon/T_j(r)], \tag{116}$$

and

$$\hat f_j=-\frac{v_\zeta}{\Omega_{\theta j}}\left[\frac{d\ln n_j}{dr}-\frac{d\ln T_j}{dr}\left(\frac{3}{2}-\frac{\epsilon}{T_j}\right)\right]. \tag{117}$$

For odd modes in the frequency range (115), the perturbed electron density amplitude is given by

$$\begin{aligned}\tilde n_{em}&\equiv\int d^3v\,\tilde f_e\\ &\equiv\int d^3v\,f_e^1\exp\{+i\omega t-in^0[\zeta-q(r)\theta]-iS(r)F(\theta)\}\\ &=\frac{en_e}{T_e}\tilde\phi_{m0,n0}(\theta,S)\ ,\end{aligned} \tag{118}$$

where f_j^1 is the perturbed particle distribution function. Correspondingly, the fluid approximation[7] may be adopted for the impurity ions. Thus we have, from the particle conservation equation,

$$-i\omega\tilde n_{Im}+i\frac{m^0c}{r_0B}\tilde\phi_{m0,n0}(\theta,S)\frac{dn_I}{dr}+n_IB\frac{\partial}{\partial l}\left(\frac{1}{B}\tilde u_{Im_\parallel}\right)=0\ , \tag{119}$$

and, from momentum conservation along the magnetic field,

$$-i\omega m_In_I\tilde u_{Im_\parallel}=-eZn_I\frac{\partial}{\partial l}\tilde\phi_{m0,n0}(\theta,S)-n_I\frac{\partial\tilde T_{Im}}{\partial l}\ , \tag{120}$$

and, from the energy balance equation,

$$-i\omega\tilde T_{Im}+i\frac{m^0c}{r_0B}\tilde\phi_{m0,n0}(\theta,S)\frac{dT_I}{dr}=0\ . \tag{121}$$

Thus

$$\tilde n_{Im}=-n_I\frac{e}{T_i}\left[\frac{\omega_{*I}}{\omega}+Z\frac{T_i}{m_I}\frac{B}{\omega^2}\left(1-\frac{\omega_{TI}}{Z\omega}\right)\frac{\partial}{\partial l}\frac{1}{B}\frac{\partial}{\partial l}\right]\tilde\phi_{m0,n0}(\theta,S)\ , \tag{122}$$

where $\omega_{*I}=-n^0cT_i(d\ln n_I/dr)/(eR_0B_\theta^0)=-m^0cT_i(d\ln n_I/dr)/(er_0B_0)$ and $\omega_{TI}=-m^0c(dT_I/dr)/(er_0B_0)$. We have defined $\partial/\partial l\equiv(\mathbf{B}\cdot\nabla)/B=(qR_0)^{-1}[(\partial/\partial\theta)+iSG(\theta)]$, when operating on functions whose spatial dependence is of the form (114). For the choice $G(\theta)=2\pi\delta(\theta\pm\pi)$ that we have made, this is $(\partial/\partial l)=(qR_0)^{-1}(\partial/\partial\theta)$.

In order to obtain the perturbed main ion density amplitude it is convenient to employ the decompositions[10]

$$\tilde\phi_{m0,n0}(\theta,S)=\sum_p\tilde\Phi^{(p)}(\Lambda,S)\exp[ip\omega_b\hat t(\theta)] \tag{123}$$

for trapped particles, and

$$\tilde\phi_{m0,n0}(\theta,S)\exp\{iS[F(\theta)-\omega_t\hat t(\theta)]\}=\sum_p\tilde\Phi^{(p)}(\Lambda,S)\exp[ip\omega_t\hat t(\theta)] \tag{124}$$

for circulating particles, where

$$\hat t(\theta)=qR_0\int_0^\theta d\theta/v_\parallel(\theta)+\text{const}\ , \tag{125}$$

and we neglect all finite gyroradius and finite banana-width effects, as well as the effects of magnetic curvature drifts. We shall employ corresponding decompositions for the perturbed distribution function amplitude

$$\begin{aligned}\tilde f_i(\theta,S,\mathbf{v})&\equiv f_i^1\exp\{+i\omega t-in^0[\zeta-q(r)\theta]-iS(r)F(\theta)\}\\ &=\sum_p\tilde f_i^{(p)}(\Lambda,\epsilon,S)\exp[ip\omega_b\hat t(\theta)]\end{aligned} \tag{126}$$

for trapped particles, and

$$\tilde{f}_i(\theta,S,\mathbf{v})\exp\{iS[F(\theta)-\omega_t\hat{t}(\theta)]\} = \sum_p \tilde{f}_i^{(p)}(\Lambda,\epsilon,S)\exp[ip\omega_t\hat{t}(\theta)] \quad (127)$$

for circulating particles. All of these summations are over all integer values of p.

For the modes under consideration, we may adopt the guiding center approximation, so that the relevant kinetic equation is[11]

$$\frac{\partial}{\partial t}f_i^1 + v_\parallel\nabla_\parallel f_i^1 + \frac{c}{B^2}(\tilde{\mathbf{E}}\times\mathbf{B})\cdot\frac{\partial}{\partial\mathbf{x}}f_i^0 + \frac{e}{m_i}\tilde{E}_\parallel\frac{\partial}{\partial v_\parallel}f_i^0 = 0\ , \quad (128)$$

and, from this, we obtain

$$(-i\omega + v_\parallel\nabla_\parallel)f_i^1 = -\frac{e}{T_i}f_{Mi}[v_\parallel\nabla_\parallel - i\omega_{*i}^T(\epsilon)]\tilde{\Phi}\ , \quad (129)$$

where $\omega_{*i}^T(\epsilon) = \omega_{*i}[1-\eta_i(\frac{3}{2}-\epsilon/T_i)]$, $\omega_{*i} = -m^0cT_i(d\ln n_i/dr)/(er_0B_0) = -n^0cT_i(d\ln n_i/dr)/(eR_0B_\theta^0)$, and $\eta_i = (d\ln T_i/dr)/(d\ln n_i/dr)$.

We note that

$$v_\parallel\nabla_\parallel \simeq \dot{\theta}\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\hat{t}}\ . \quad (130)$$

Therefore, if we introduce the decompositions (123), (124), (126), and (127), we obtain

$$(-i\omega + ip\omega_b)\tilde{f}_i^{(p)} = -i\frac{e}{T_i}f_{Mi}[p\omega_b - \omega_{*i}^T(\epsilon)]\tilde{\Phi}^{(p)} \quad (131)$$

for trapped particles, and

$$[-i\omega + i(p+S)\omega_t]\tilde{f}_i^{(p)} = -i\frac{e}{T_i}f_{Mi}[(p+S)\omega_t - \omega_{*i}^T(\epsilon)]\tilde{\Phi}^{(p)} \quad (132)$$

for circulating particles. Thus,

$$\tilde{f}_i = \frac{e}{T_i}f_{Mi}\sum_p \tilde{\Phi}^{(p)}\exp[ip\omega_b\hat{t}(\theta)]\frac{p\omega_b - \omega_{*i}^T}{\omega - p\omega_b} \quad (133)$$

for trapped particles. Writing $p\omega_b - \omega_{*i}^T = -(\omega - p\omega_b) + (\omega - \omega_{*i}^T)$ and using (123) this is

$$\tilde{f}_i = -\frac{e}{T_i}f_{Mi}\left(\tilde{\phi}_{m0,n0}(\theta,S) - \sum_p \tilde{\Phi}^{(p)}\exp[ip\omega_b\hat{t}(\theta)]\frac{\omega - \omega_{*i}^T(\epsilon)}{\omega - p\omega_b}\right). \quad (134)$$

Similarly, using (128), we have for circulating particles

$$\tilde{f}_i = -\frac{e}{T_i}f_{Mi}\left(\tilde{\phi}_{m0,n0}(\theta,S) - \sum_p \tilde{\Phi}^{(p)}\exp[ip\omega_t\hat{t}(\theta)] \times\exp\{-iS[F(\theta)-\omega_t\hat{t}(\theta)]\}\frac{\omega - \omega_{*i}^T(\epsilon)}{\omega - (p+S)\omega_t}\right)\ . \quad (135)$$

Now, we consider the quadratic form

$$\begin{aligned}
2\pi qR_0\langle\langle\tilde{\phi}_{m0,n0}^*(\theta,S)\tilde{n}_{im}\rangle\rangle &\equiv qR_0\int_{-\pi}^{\pi}d\theta\, h(\theta)\int_{-1/2}^{1/2}dS\,\tilde{\phi}_{m0,n0}^*(\theta,S)\tilde{n}_{im} \\
&= \frac{\pi}{2}\left(\frac{2}{m_i}\right)^2\sum_\sigma\int_0^\infty d\epsilon\,\epsilon\int_0^{1+\epsilon_0}d\Lambda\int_{-1/2}^{1/2}dS\int_{-\theta_0}^{\theta_0}d\theta\frac{1}{|\dot{\theta}|}\tilde{\phi}_{m0,n0}^*(\theta,S)\tilde{f}_i \\
&= -\frac{en_i}{T_i}\Bigg[qR_0\int_{-\pi}^{\pi}d\theta\, h(\theta)\int_{-1/2}^{1/2}dS\,|\tilde{\phi}_{m0,n0}(\theta,S)|^2 - \frac{\pi}{2n_i}\left(\frac{2}{m_i}\right)^2\sum_\sigma\int_{-1/2}^{1/2}dS\int_0^\infty d\epsilon\,\epsilon f_{Mi}[\omega - \omega_{*i}^T(\epsilon)] \\
&\quad\times\left(\int_{1-\epsilon_0}^{1+\epsilon_0}d\Lambda\frac{\tau_b}{2}\sum_p\frac{|\tilde{\Phi}^{(p)}(\Lambda,S)|^2}{\omega - p\omega_b} + \int_0^{1-\epsilon_0}d\Lambda\,|\tau_t|\sum_p\frac{|\tilde{\Phi}^{(p)}(\Lambda,S)|^2}{\omega - (p+S)\omega_t}\right)\Bigg], \quad (136)
\end{aligned}$$

where $\sigma = \mathrm{sgn}(v_\parallel)$, $\theta_0 = \pi$ for circulating particles, and $\theta_0 = \arccos[(\Lambda-1)/\epsilon_0]$ for trapped particles.

In the limit $\omega < \langle\omega_b\rangle_i$ we may expand the resonant denominators in (136). For trapped particles

$$\frac{1}{\omega - p\omega_b} \simeq -\left(\frac{1}{p\omega_b} + \frac{\omega}{p^2\omega_b^2} + i\pi\delta(\omega - p\omega_b)\right)\ , \quad (137)$$

and, noting that $\omega_b(\Lambda,\epsilon) = (\epsilon/T_i)^{1/2}\hat{\omega}_b(\Lambda) \equiv X\hat{\omega}_b(\Lambda)$,

$$\delta(\omega - p\omega_b) = \frac{1}{|p|\,\hat{\omega}_b(\Lambda)}\delta\left(X - \frac{\omega}{p\hat{\omega}_b(\Lambda)}\right)\ . \quad (138)$$

Similar results hold for the circulating particles. From the expression for $\tilde{\Phi}_R^{(p)}(\Lambda,S)$ obtained from (123) for trapped particles, we see that $\tilde{\Phi}^{(0)} = 0$ for odd modes and $|\tilde{\Phi}^{(p)}|^2$ is even in the sign of p. Similarly, $|\tilde{\Phi}^{(p)}(\Lambda,S)|^2$ is even in the sign σ of $v_\parallel$ for circulating particles. Using all of these results, the quadratic form (136) may be expressed as

$$-\frac{en_i}{T_i}qR_0\left(\hat{\Pi}_0 - \frac{\omega[\omega_{*i}(1-\eta_i) - \omega]}{(\pi\overline{\omega}_{ti})^2\epsilon_0^{1/2}}\hat{\Pi}_1 - i\frac{\omega^2[\omega_{*i}(1-\frac{3}{2}\eta_i) - \omega]}{\pi^{5/2}\overline{\omega}_{ti}^3\epsilon_0}\hat{\Pi}_2\right)\ , \quad (139)$$

where

$$\hat{\Pi}_0 \equiv \int_{-\pi}^{\pi}d\theta\, h(\theta)\int_{-1/2}^{1/2}dS\,|\tilde{\phi}_{m0,n0}(\theta,S)|^2\ , \quad (140)$$

$$\hat{\Pi}_1 \equiv \epsilon_0^{-1}\int_{-1/2}^{1/2}dS\left(2\int_{1-\epsilon_0}^{1+\epsilon_0}d\Lambda\, L_b^3(\Lambda)\sum_{p=1}^{\infty}|\tilde{\Phi}^{(p)}(\Lambda,S)|^2/p^2 + \frac{1}{4}\int_0^{1-\epsilon_0}d\Lambda\, L_t^3(\Lambda)\sum_{p=-\infty}^{\infty}|\tilde{\Phi}^{(p)}(\Lambda,S)|^2/(p+S)^2\right)\ , \quad (141)$$

$$\hat{\Pi}_2 \equiv \epsilon_0^{-1}\int_{-1/2}^{1/2} dS \left(2\int_{1-\epsilon_0}^{1+\epsilon_0} d\Lambda\, L_b^4(\Lambda)\sum_{p=1}^{\infty} |\tilde{\Phi}^{(p)}(\Lambda,S)|^2/p^3 + \frac{1}{8}\int_0^{1-\epsilon_0} d\Lambda\, L_t^4(\Lambda)\sum_{p=-\infty}^{\infty} |\tilde{\Phi}^{(p)}(\Lambda,S)|^2/|p+S|^3\right), \qquad (142)$$

$$L_b(\Lambda) = \epsilon_0^{1/2}\int_{-\theta_0(\Lambda)}^{\theta_0(\Lambda)} d\theta\,[1-\Lambda/h(\theta)]^{-1/2}, \qquad (143)$$

and

$$L_t(\Lambda) = \epsilon_0^{1/2}\int_{-\pi}^{\pi} d\theta\,[1-\Lambda/h(\theta)]^{-1/2}. \qquad (144)$$

All of the $\hat{\Pi}$'s are real, positive definite quantities of zeroth order in ϵ_0 and in $\omega/\langle\omega_b\rangle_i$.

Collecting our results (118), (122), and (139), and taking $\omega < \omega_{*i}$, as can be justified *a posteriori*, we may write the quadratic form of the quasi-neutrality condition

$$\langle\langle \tilde{\phi}^*_{m0,n0}(\theta,S)(\tilde{n}_{im} + Z\tilde{n}_{Im} - \tilde{n}_{em})\rangle\rangle = 0 \qquad (145)$$

as

$$\left(\frac{T_i}{T_e}+\frac{n_i}{n_e}\right) - Z^2\frac{n_I}{n_e}\frac{m_i}{m_I}\frac{\overline{\omega}_{ti}^2}{2\omega^2}\left(1-\frac{\omega_{TI}}{Z\omega}\right)J_3 + Z\frac{n_I}{n_e}\frac{\omega_{*I}}{\omega} - \frac{n_i}{n_e}\left(\frac{\omega(\omega_{*i}-\omega_{Ti})}{\epsilon_0^{1/2}\overline{\omega}_{ti}^2}J_1 + i\frac{\omega^2(\omega_{*i}-\frac{3}{2}\omega_{Ti})}{\epsilon_0\overline{\omega}_{ti}^3}J_2\right) = 0, \qquad (146)$$

where $\omega_{Ti} \equiv \eta_i\omega_{*i}$, $J_1 = \hat{\Pi}_1/(\pi^2\hat{\Pi}_0)$, $J_2 = \hat{\Pi}_2/(\pi^{5/2}\hat{\Pi}_0)$, and

$$J_3 = \int_{-\pi}^{\pi} d\theta\, h(\theta)\int_{-1/2}^{1/2} dS\left|\frac{\partial}{\partial\theta}\tilde{\phi}_{m0,n0}(\theta,S)\right|^2 \times\left(\int_{-\pi}^{\pi} d\theta\, h(\theta)\int_{-1/2}^{1/2} dS\,|\tilde{\phi}_{m0,n0}(\theta,S)|^2\right)^{-1}. \qquad (147)$$

All of the J's are real, dimensionless, positive quantities of order one.

In deriving (146), we have neglected the effects of magnetic-curvature drifts for the main ions and impurity ions. Including this effect for the impurity ions leads to the additional term[10]

$$Z\frac{n_I}{n_e}\frac{\overline{\omega}_{DI}}{\omega}J_5\left(1-\frac{\omega_{*I}}{\omega}\frac{T_I}{ZT_i}(1+\eta_I)\right),$$

where $\overline{\omega}_{DI} \equiv \omega_{*I}(r_{nI}/r_0)\epsilon_0$ and

$$J_5 = 2\int_{-\pi}^{\pi} d\theta\, h(\theta)\cos\theta\int_{-1/2}^{1/2} dS\,|\tilde{\phi}_{m0,n0}(\theta,S)|^2 \times\left(\int_{-\pi}^{\pi} d\theta\, h(\theta)\int_{-1/2}^{1/2} dS\,|\tilde{\phi}_{m0,n0}(\theta,S)|^2\right)^{-1}.$$

Here, we have integrated over the impurity distribution function the magnetic-curvature drift frequency $\omega_{DI} = (m^0/r_0)v_{DI}Z$, where v_{DI} is the poloidal component of the magnetic-curvature drift velocity. It may be seen that this term is of higher order in ϵ_0 than the third term in (146), and may be neglected. The effect of main ion magnetic-curvature drifts is even less important.

B. Special cases and mode classification

Equation (146) is the general quadratic form for *odd* modes in the frequency range (115). When the impurity temperature gradient is small enough to neglect, it is a fourth order polynomial in ω; of the four roots, usually at most one has both $\mathrm{Im}\omega > 0$ and $|\omega| < \langle\omega_b\rangle_i$. When the parameters are such that the impurity sound term in J_3 is one of the dominant terms in (146) for this root, we label it an impurity sound mode, otherwise, it is labeled an impurity (trapped particle) mode. When the relative impurity temperature gradient is large compared with the relative impurity density gradient, an impurity temperature gradient mode form of the impurity sound mode can arise due to the impurity sound temperature gradient term, with frequency $\omega \gtrsim \langle\omega_t\rangle_I$. All of these modes are nonresonant, in that they can be unstable even without consideration of the mode-particle resonance term in J_2. Later in this section, we shall repeat the consideration of these modes several times in successivley more comprehensive fashion.

All of the modes mentioned so far have a perturbed electrostatic potential which is *odd* about the point of minimum magnetic field. For *even* modes, the corresponding quadratic form would contain important additional terms. In the present collisionless limit, there are no significant even modes which arise specifically because of the presence of impurities. Including the relevant terms due to collisions, however, new modes can arise due to impurities, and modifications can be caused by impurities in modes which would be unstable even without impurities. These modes will be discussed more specifically in Sec. VI.

We now want to consider various special cases for the quadratic form (146) before treating it more generally. We first consider the case where the main ion and the impurity ion temperature gradients are negligible. We take $\omega\omega_{*i} \sim \overline{\omega}_{ti}^2\epsilon_0^{1/2}$ and $T_i \sim T_e$. Equation (146) becomes

$$\left(\frac{T_i}{T_e}+\frac{n_i}{n_e}\right) - Z_e\frac{n_i}{n_e}\frac{m_i}{m_I}\frac{\overline{\omega}_{ti}^2}{2\omega^2}J_3 + Z\frac{n_I}{n_e}\frac{\omega_{*I}}{\omega} - \frac{n_i}{n_e}\left(\frac{\omega_{*i}\omega}{\epsilon_0^{1/2}\overline{\omega}_{ti}^2}J_1 + i\frac{\omega_{*i}\omega^2}{\epsilon_0\overline{\omega}_{ti}^3}J_2\right) = 0. \qquad (148)$$

Now, we can refer to (148) in order to find conditions under which unstable modes can be found. We identify two types which can be unstable with $\mathrm{Im}\omega = \gamma \gtrsim \mathrm{Re}\omega = \omega_0$, and which therefore do not require consideration of the term in J_2. One kind of mode corresponds to the case where the terms in J_1 and J_3 are largest within (148), and does not depend on the relative sign of the impurity ion density gradient versus that of the main ion density gradient. We can consider this as the trapped particle version of the impurity sound mode. In particular, the condition that the second term in (148) dominate the first term, along with (115), requires that

$$Z_e \gtrsim \left(\frac{\omega}{\bar{\omega}_{tI}}\right)^2 \frac{T_I}{T_i}(2J_3) \ . \tag{149}$$

Then, we obtain

$$\omega^3 \simeq -Z_e \frac{m_i}{m_I} \frac{\epsilon_0^{1/2}\bar{\omega}_{ti}^4}{\omega_{*i}} \frac{J_3}{2J_1} \ . \tag{150}$$

Finally, the requirement $\omega < \langle\omega_b\rangle_i$ implies that

$$\omega_{*i} > \left(Z_e \frac{m_i}{m_I \epsilon_0}\right)\bar{\omega}_{ti}\left(\frac{J_3}{2^{5/2}J_1}\right),$$

and this sets a lower limit on the values of m^0 and $(r_{ni}/r_0)(m_i Z_e/m_I)/q_0$ that can be consistently considered.

Now, if we consider a situation where the impurity sound term can be neglected in Eq. (148), such as when $Z_e \sim 1$, and retain the lowest order terms in $\omega/\langle\omega_b\rangle_i$, we have a quadratic equation whose solutions are, for $n_i \simeq n_e$,

$$\omega = \frac{\epsilon_0^{1/2}\bar{\omega}_{ti}^2}{2\omega_{*i}J_1}\left\{\left(\frac{T_i}{T_e}+1\right) \pm \left[\left(\frac{T_i}{T_e}+1\right)^2 + 4\frac{Z_e}{Z}\frac{\omega_{*I}\omega_{*i}J_1}{\epsilon_0^{1/2}\bar{\omega}_{ti}^2}\right]^{1/2}\right\} \ . \tag{151}$$

Then, we have a nonresonant instability, since it does not depend on the term in J_2, if

$$\frac{Z_e}{Z}\sigma_I < -\frac{\epsilon_0^{1/2}}{8b_i J_1}\left(1+\frac{T_i}{T_e}\right)^2\left(\frac{r_{ni}}{qR_0}\right)^2 , \tag{152}$$

where $b_i \equiv \frac{1}{2}(m^0/r)^2\rho_i^2$ and $\rho_i = v_i/\Omega_i$, implying that, in this case, impurity ions and main ions should have opposite density gradients, as can be realized when the impurity ions are concentrated in the outer region of the plasma column. When the inequality (152) is not satisfied and $\omega_0 \simeq -(Z_e/Z)\omega_{*I}/(1+T_i/T_e)$, we have the impurity drift mode whose (resonant) growth rate is given by

$$\gamma = \omega_{*i}\frac{\omega_0^3}{\bar{\omega}_{ti}^3}\frac{J_2}{\epsilon_0(1+T_i/T_e)} , \tag{153}$$

and is positive for $\sigma_I < 0$.

It is important to consider the effects of the ion temperature gradient, and for this we refer to Eq. (146). The equivalent of Eq. (151) is

$$\omega = \frac{\epsilon_0^{1/2}\bar{\omega}_{ti}^2}{2\omega_{*i}(1-\eta_i)J_1}\left\{\left(1+\frac{T_i}{T_e}\right) \pm \left[\left(1+\frac{T_i}{T_e}\right)^2 + 4\frac{Z_e}{Z} \times \frac{\omega_{*I}\omega_{*i}}{\epsilon_0^{1/2}\bar{\omega}_{ti}^2}J_1(1-\eta_i)\right]^{1/2}\right\}, \tag{154}$$

and the condition for nonresonant instability is now

$$\frac{Z_e}{Z}\sigma_I(\eta_i - 1) > \frac{\epsilon_0^{1/2}}{8b_i J_1}\left(1+\frac{T_i}{T_e}\right)^2\left(\frac{r_{ni}}{qR_0}\right)^2 . \tag{155}$$

When this condition is not satisfied and instead we obtain the usual frequency of the impurity drift mode, the growth rate γ is given by, instead of Eq. (153),

$$\gamma = \omega_{*i}\left(\frac{\omega_{*i}Z_e}{\bar{\omega}_{ti}Z(1+T_i/T_e)}\right)^3 \sigma_I^3(\tfrac{3}{2}\eta_i - 1)\frac{J_2}{\epsilon_0(1+T_i/T_e)} \ . \tag{156}$$

Therefore, we obtain instability even with $\sigma_I > 0$ if only

$$\eta_i > \tfrac{2}{3} ,$$

which is an easy condition to satisfy in realistic situations. When the condition (155) for a nonresonant instability is satisfied, and

$$\omega \sim \left(\frac{Z_e}{Z}\epsilon_0^{1/2}\sigma_I J_1\right)^{1/2}\bar{\omega}_{ti} , \tag{157}$$

we require

$$\frac{m_i}{m_I}\frac{1}{\epsilon_0^{1/2}} < \frac{Z_e}{Z}\sigma_I < \epsilon_0^{1/2} \tag{158}$$

for consistency with the assumed frequency range (115).

In order to investigate the regions in parameter space corresponding to the instability with $\omega < \langle\omega_b\rangle_i$ more fully, the fifth-order polynomial equation in ω, (146), has been solved numerically. For the sake of definiteness we have chosen the illustrative values $J_1 = J_2 = J_3 = 1$, $T_i/T_e = 0.5$, $\epsilon_0 = 0.01$, and $m_I/m_i = Z$. We note that, from the neutrality condition, we have $Zn_I/n_e = 1-(n_i/n_e)$, so that $0 \le (Zn_I/n_e) \le 1$. In general, it is seen from the numerical solutions of Eq. (146) that, for small concentrations of impurities, the conditions for acceptable instability are less stringent than would be suggested by the special cases treated earlier, and, for larger concentrations of impurities, that the same conditions on the relative directions of the main ion and impurity ion density gradients apply as were deduced for small concentrations of impurities.

If an impurity ion temperature gradient were included with $\eta_I \equiv (d\ln T_I/dr)/(d\ln n_I/dr) > 1$, then an instability with $\langle\omega_t\rangle_I \lesssim \omega < \langle\omega_b\rangle_i \ll \langle\omega_b\rangle_e$ could occur, with the main ions and electrons responding adiabatically. This is analogous to the instability found in the frequency range $\langle\omega_t\rangle_i \lesssim \omega < \langle\omega_b\rangle_e$ with $\eta_i > 1$ in the absence of impurities. For this mode the J_1 and J_2 terms in (146) may be neglected. The resulting cubic equation in ω always has one unstable root. The simplest form of this root is seen by keeping only the first, second, and fourth terms in (146), so that

$$\omega^3 \simeq -Z\frac{n_I}{n_e}\frac{m_i}{2m_I}\bar{\omega}_{ti}\omega_{TI}\left(\frac{T_i}{T_I}+\frac{n_i}{n_e}\right)^{-1} . \tag{159}$$

We label this as the impurity temperature gradient sound mode, or as the impurity temperature gradient mode, since it is, in fact, an extension of the impurity sound mode discussed previously. We also recall from Sec. IV that this mode can play an important role in the evolution from σ_I positive to negative.

C. Quasi-linear estimates

We notice that $\tilde{n}_{em}$ and $\tilde{\phi}_{m0,n0}(\theta, S)$ are in phase, as indicated by (118). Therefore, no net transport of electrons across the magnetic field is found to be produced by these modes when the evolution of the average distribution function is estimated by the well-known quasilinear theory. On the basis of the same theory, we also expect that a rearrangement of the impurity and main ion spatial distributions, as well as transport of ion thermal energy, across the magnetic field does result. Since quasi-linear theory is not entirely appropriate for dealing with nonresonant instabilities, such as the ones that we have found, we shall merely use it for reasonable estimates of the produced transport.

In order to estimate the impurity particle flux that can result from these modes, as well as the evolution of the relevant equilibrium, we consider the quasi-linear equation for the main ion density evolution as we did in Sec. IV.

$$\frac{\partial}{\partial t} n_i + \langle\langle \nabla \cdot (\tilde{\mathbf{v}}_E n_i^1 + \text{c.c.} \rangle\rangle = 0 \ , \qquad (160)$$

where $\tilde{\mathbf{v}}_E = -c\nabla\tilde{\Phi} \times \mathbf{B}/B^2$ and $n_i^1 = \tilde{n}_{im} \exp\{-i\omega t + in^0[\zeta - q(r)\theta] + iS(r)F(\theta)\}$. Now, it is easy to see that $\langle\langle \nabla \cdot (\tilde{\mathbf{v}}_E n_i^1 + \text{c.c.})\rangle\rangle = (\partial/\partial r)\langle\langle \tilde{v}_{E r} n_i^1 + \text{c.c.}\rangle\rangle$, where the radial derivative is taken over a scale longer than Δr_s. The resulting main ion flux is then

$$\Gamma_i = 2\sum_{m^0} \frac{cm^0}{r_0 B_0} \left\langle\left\langle |\tilde{\Phi}|^2 \ \text{Im}\left(\frac{n_i^1}{\tilde{\Phi}}\right)\right\rangle\right\rangle . \qquad (161)$$

Similarly, we may consider the quasi-linear main ion thermal energy transport across the magnetic field due to these modes. To estimate this we evaluate

$$\frac{\partial}{\partial t} W_{\perp i} + \langle\langle \nabla \cdot (\tilde{\mathbf{v}}_E \tilde{W}_{\perp i} + \text{c.c.})\rangle\rangle = 0 \qquad (162)$$

for the main ions, where $W_{\perp i} \equiv (m_i/2)\int d^3\mathbf{v}\, v_\perp^2 f_i^0$ and $\tilde{W}_{\perp i} \equiv (m_i/2)\int d^3\mathbf{v}\, v_\perp^2 f_i^1$. We see that $\langle\langle \nabla \cdot (\tilde{v}_E \tilde{W}_{\perp i} + \text{c.c.})\rangle\rangle = (\partial/\partial r)\langle\langle \tilde{\mathbf{v}}_{E r} \tilde{W}_{\perp i} + \text{c.c.}\rangle\rangle$, as before. The resulting heat flux is

$$q_{\perp i} = 2\sum_{m^0} \frac{m^0 c}{r_0 B_0} \langle\langle |\tilde{\Phi}|^2 \ \text{Im}(\tilde{W}_{\perp i}/\tilde{\Phi})\rangle\rangle \ . \qquad (163)$$

Now, as in Sec. IV, we may recall the quasi-neutrality condition (145) and obtain

$$\Gamma_I = -(1/Z)\Gamma_i \qquad (164)$$

from

$$\Gamma_i = \frac{2}{\pi r_{ni}} n_i D_{Bi} \sum_{m^0} \left|\frac{e\tilde{\Phi}}{T_i}\right|^2 k_\theta^2 \rho_i \frac{qR_0}{2} \times\left\{\frac{J_2}{\overline{\omega}_{ti}^2 \epsilon_0}\left[(\omega_0^2 - \gamma^2)(1 - \tfrac{3}{2}\eta_i) - (\omega_0^2 - 3\gamma^2)\frac{\omega_0}{\omega_{*i}}\right] + \frac{\gamma}{\overline{\omega}_{ti}\epsilon_0^{1/2}} J_1 \left(1 - \eta_i - 2\frac{\omega_0}{\omega_{*i}}\right)\right\} , \qquad (165)$$

where $D_{Bi} = cT_i/(eB_0)$ and $\omega = \omega_0 + i\gamma$. Evidently, both terms in (165) will contribute to the resonant instabilities and the second term will dominate for nonresonant instabilities.

Taking the root corresponding to the resonant impurity drift mode (156) for ω, so that $\omega_0^2 > \gamma^2$, we find that the impurity flux is outgoing if

$$\eta_i > \left[\left(1 - \frac{\omega_0}{\omega_{*i}}\right) - \frac{\omega_0^2 J_1}{\overline{\omega}_{ti}^2 \epsilon_0^{1/2}} \frac{Z}{Z_e \sigma_I}\left(1 - 2\frac{\omega_0}{\omega_{*i}}\right)\right] \times\left(\frac{3}{2} - \frac{\omega_0^2 J_1}{\overline{\omega}_{ti}^2 \epsilon_0^{1/2}} \frac{Z}{Z_e \sigma_I}\right)^{-1} . \qquad (166)$$

When $[(\omega/\langle\omega_b\rangle_i)^2 \epsilon_0^{1/2} Z/(Z_e\sigma_I)] < 1$ and $\omega_0 < \omega_{*i}$, this condition reduces to $\eta_i > \frac{2}{3}$. The impurity flux Γ_I will go to zero if

$$\left[1 - \tfrac{3}{2}\eta_i + \frac{Z_e}{Z}\sigma_I\left(1 + \frac{T_i}{T_e}\right)^{-1}\right] - \frac{\omega_0^2 J_1}{\overline{\omega}_{ti}^2 \epsilon_0^{1/2}} \times \frac{Z}{Z_e\sigma_I}\left[1 - \eta_i + 2\frac{Z_e}{Z}\sigma_I\left(1 + \frac{T_i}{T_e}\right)^{-1}\right] = 0 \ . \qquad (167)$$

For the nonresonant form (164) of the impurity drift mode, the impurity flux is outgoing if

$$0 > 1 - \eta_i - \frac{\epsilon_0^{1/2}}{2J_1(1-\eta_i)b_i}\left(\frac{r_{ni}}{qR_0}\right)^2 , \qquad (168)$$

and Γ_I will vanish if the quantity on the right-hand side of (168) equals zero. In the case of the (nonresonant) impurity sound mode (150), the impurity flux is positive if

$$\eta_i > 1 - 2\left(Z_e \frac{m_i}{m_I} \frac{\epsilon_0^{1/2}\overline{\omega}_{ti}^4}{\omega_{*i}^4} \frac{J_3}{2J_1}\right)^{1/3} . \qquad (169)$$

Turning to the question of the transport of main ion thermal energy, the result for $\tilde{W}_{\perp i}$ corresponding to (139) for $\tilde{n}_i$ is

$$2\pi qR_0 \langle\langle \tilde{\phi}_{m^0,n^0}(\theta,S)\tilde{W}_{\perp i}\rangle\rangle = -en_i qR_0\left(\hat{\Pi}_0 - \frac{\omega(\omega_{*i} - \omega)}{(\pi\overline{\omega}_{ti})^2\epsilon_0^{1/2}} 2\hat{\Pi}_1' - i\frac{\omega^4\{\omega_{*i}[1 - \frac{3}{2}\eta_i] - \omega\}}{\pi^{9/2}\overline{\omega}_{ti}^5\epsilon_0^2} 2\hat{\Pi}_2'\right), \qquad (170)$$

where

$$\hat{\Pi}_1' \equiv \epsilon_0^{-1}\int_{-1/2}^{1/2} dS\left[2\int_{1-\epsilon_0}^{1+\epsilon_0} d\Lambda\, L_b^3(\Lambda)\Lambda \sum_{p=1}^{\infty} |\tilde{\Phi}^{(p)}(\Lambda,S)|^2/p^2 + \frac{1}{4}\int_0^{1-\epsilon_0} d\Lambda\, L_t^3(\Lambda)\Lambda \sum_{p=-\infty}^{\infty} |\tilde{\Phi}^{(p)}(\Lambda,S)|^2/(p+S)^2\right], \qquad (171)$$

and

$$\hat{\Pi}_2' \equiv \epsilon_0^{-1}\int_{-1/2}^{1/2} dS\left[2\int_{1-\epsilon_0}^{1+\epsilon_0} d\Lambda\, L_b^6(\Lambda)\Lambda \sum_{p=1}^{\infty} |\tilde{\Phi}^{(p)}(\Lambda,S)|^2/p^5 + \frac{1}{32}\int_0^{1-\epsilon_0} d\Lambda\, L_t^6(\Lambda)\Lambda \sum_{p=-\infty}^{\infty} |\tilde{\Phi}^{(p)}(\Lambda,S)|^2/|p+S|^5\right]. \qquad (172)$$

Then, the expression (163) for the main ion thermal energy flux becomes

$$q_{\perp i} = \frac{2}{\pi r_{Ti}} n_i T_i D_{Bi} \sum_{m^0} \left|\frac{e\tilde{\Phi}}{T_i}\right|^2 \frac{k_\theta^2 \rho_i}{\eta_i} \frac{qR_0}{2}\left\{\frac{J_1'}{\overline{\omega}_{ti}\epsilon_0^{1/2}}\left(1 - 2\frac{\omega_0}{\omega_{*i}}\right) + \frac{J_2'}{\overline{\omega}_{ti}^4\epsilon_0^2}\left[(\omega_0^4 - 6\gamma^2\omega_0^2 + \gamma^4)(1 - \tfrac{3}{2}\eta_i) - (\omega_0^4 - 10\gamma^2\omega_0^2 + 5\gamma^4)\frac{\omega_0}{\omega_{*i}}\right]\right\}, \qquad (173)$$

where $J_1' \equiv 2\hat{\Pi}_1'/(\pi^2\hat{\Pi}_0)$, $J_2' = 2\hat{\Pi}_2'/(\pi^{9/2}\hat{\Pi}_0)$, and $r_{Ti} \equiv -(d\ln T_i/dr)^{-1}$. The second term in (173) will be negligible for $\omega < \langle\omega_b\rangle_i$. Thus, since $\omega_0 < \omega_{*i}$ for the modes under consideration, we see that the ion heat transport is always outward. If we define an impurity "diffusion" coefficient and an ion thermal diffusion coefficient as in Sec. IV, we can see that again

$$D_I \sim (Z/Z_e) D_{Ti} \ . \qquad (174)$$

Thus, the rate of change of n_I can be expected to be faster than that of either n_i or T_i, as for the modes discussed in Sec. IV. We therefore again expect that the evolution of σ_I will be such as to stabilize the various impurity modes, with η_i little affected on the relevant time scale.

VI. COLLISIONAL KINETIC MODES

In this section, we consider the effects of collisions which were neglected in Sec. V. We refer to the frequency range

$$\langle\omega_t\rangle_I<\omega<\nu_{iI}/\epsilon_0<\langle\omega_b\rangle_i\ll\langle\omega_b\rangle_e\,, \tag{175}$$

where we assume that

$$\nu_{iI}\gtrsim\nu_{ii}\,. \tag{176}$$

We consider modes with $\tilde{\phi}_m(\theta)$ *odd* and notice that, from the results in Sec. V,

$$\tilde{n}_{em}=\frac{en_e}{T_e}\tilde{\phi}_{m^0,n^0}(\theta,S)\,, \tag{177}$$

and

$$\tilde{n}_{Im}\simeq-\frac{en_I}{T_i}\frac{\omega_{*I}}{\omega}\tilde{\phi}_{m^0,n^0}(\theta,S)\,, \tag{178}$$

if we maintain the ordering

$$Z^2\frac{n_I}{n_i}\sim1,\quad T_I\sim T_i\lesssim T_e,\quad\frac{\omega_{*I}}{\omega}\sim Z>1\,. \tag{179}$$

If instead we can have the ordering

$$1<Z^2\frac{n_I}{n_i}<\frac{m_I}{m_i}\epsilon_0\,, \tag{180}$$

and consider

$$\frac{\omega_{*I}}{Z\omega}\lesssim\frac{m_i}{m_I}\frac{\bar{\omega}_{ti}^2}{2\omega^2}\,, \tag{181}$$

the impurity sound term should also be retained, as in Eq. (122).

Now, we consider modes for which

$$\frac{\partial\tilde{f}_i}{\partial t}\ll\left(\frac{\delta\tilde{f}_i}{\delta t}\right)_c\,, \tag{182}$$

the effects of finite main ion gyroradii, banana widths, and magnetic curvature drifts can be neglected, and the guiding center approximation can be adopted, as in Sec. V. We also consider a simplified collision operator representing the pitch angle scattering of both trapped and barely circulating particles, so that

$$\left(\frac{\delta\tilde{f}_i}{\delta t}\right)_c\simeq-\nu_{ef}(\epsilon)\left(\tilde{f}_i+\frac{e\tilde{\phi}}{T_i}f_{Mi}\right)\,, \tag{183}$$

where

$$\nu_{ef}(\epsilon)=\frac{\nu_{iI}}{\epsilon_0}\left(\frac{T_i}{\epsilon}\right)^{3/2}\,, \tag{184}$$

and we have corrected the collision term for the Maxwellian part of $\tilde{f}_i$. As shall be seen *a posteriori*, the most important collisional contribution to the mode growth rate is, in fact, due to trapped and weakly circulating ions. For this case the relevant kinetic equation becomes, instead of (128),

$$\frac{\partial f_i^1}{\partial t}+v_\parallel\nabla_\parallel f_i^1+\frac{c}{B^2}(\tilde{\mathbf{E}}\times\mathbf{B})\cdot\frac{\partial}{\partial\mathbf{x}}f_i^0+\frac{e}{m_i}\tilde{E}_\parallel\frac{\partial}{\partial v_\parallel}f_i^0=-\nu_{ef}(\epsilon)\left(f_i^1+\frac{e\tilde{\phi}}{T_i}f_{Mi}\right) \tag{185}$$

and, from this, we obtain

$$[-i\omega+\nu_{ef}(\epsilon)+v_\parallel\nabla_\parallel]f_i^1=-(e/T_i)f_{Mi}[v_\parallel\nabla_\parallel-i\omega_{*i}^T(\epsilon)+\nu_{ef}]\tilde{\phi}\,. \tag{186}$$

Again introducing the decompositions (123), (124), (126), and (127), we obtain

$$[-i\omega+\nu_{ef}(\epsilon)+ip\omega_b]\tilde{f}_i^{(p)}=-i(e/T_i)f_{Mi}[p\omega_b-\omega_{*i}^T(\epsilon)-i\nu_{ef}]\tilde{\Phi}^{(p)} \tag{187}$$

for trapped ions, and

$$[-i\omega+\nu_{ef}(\epsilon)+i(p+S)\omega_t]\tilde{f}_i^{(p)}=-i(e/T_i)f_{Mi}[(p+S)\omega_t-\omega_{*i}^T(\epsilon)-i\nu_{ef}]\tilde{\Phi}^{(p)} \tag{188}$$

for circulating particles. Thus, for the present case the quadratic form (136) changes to

$$\begin{aligned}qR_0\int_{-\pi}^{\pi}d\theta\,h(\theta)\int_{-1/2}^{1/2}dS\,\tilde{\phi}^*_{m^0,n^0}(\theta,S)\,\tilde{n}_{im}=-\frac{en}{T_i}\Bigg[&qR_0\int_{-\pi}^{\pi}d\theta\,h(\theta)\int_{-1/2}^{1/2}dS\,|\tilde{\phi}_{m^0,n^0}(\theta,S)|^2-\frac{\pi}{n_i}\left(\frac{2}{m_i}\right)^2\sum_\sigma\int_{-1/2}^{1/2}dS\\&\times\int_0^\infty d\epsilon\,\epsilon f_{Mi}[\omega-\omega_{*i}^T(\epsilon)]\int_{1-\epsilon_0}^{1+\epsilon_0}d\Lambda\,\frac{\tau_b}{2}\sum_p\frac{|\tilde{\Phi}^{(p)}(\Lambda,S)|^2}{\omega+i\nu_{ef}(\epsilon)-p\omega_b}+\int_{1-\epsilon_0-\delta_0}^{1-\epsilon_0}d\Lambda\,|\tau_t|\sum_p\frac{|\tilde{\Phi}^{(p)}(\Lambda,S)|^2}{\omega+i\nu_{ef}(\epsilon)-(p+S)\omega_t}\\&+\int_0^{1-\epsilon_0-\delta_0}d\Lambda\,|\tau_t|\sum_p\frac{|\tilde{\Phi}^{(p)}(\Lambda,S)|^2}{\omega+i\nu_{iI}(T_i/\epsilon)^{3/2}-(p+S)\omega_t}\Bigg]\,,\end{aligned} \tag{189}$$

where δ_0 is a positive constant of order ϵ_0. We have taken $\nu_{ef}\sim\nu_{iI}/\epsilon_0$ for $\Lambda>1-\epsilon_0-\delta_0$ and $\nu_{ef}\sim\nu_{iI}$ for $\Lambda<1-\epsilon_0-\delta_0$. In the limit $(\nu_{iI}/\epsilon_0)<\langle\omega_b\rangle_i$ we may perform the energy integration by expanding the denominators in (189) separately for $0\le X\equiv(\epsilon/T_i)^{1/2}\lesssim[\nu_{iI}/(\epsilon_0p\hat{\omega}_b(\Lambda))]^{1/4}$ and for $[\nu_{iI}/(\epsilon_0p\hat{\omega}_b(\Lambda))]^{1/4}\lesssim X$, recalling that $\omega_b(\epsilon,\Lambda)=\hat{\omega}_b(\Lambda)X$ and that $\nu_{ef}(\epsilon)=(\nu_{iI}/\epsilon_0)X^{-3}$ for trapped particles. Analogous expressions apply for circulating particles. Thus, the quadratic form (139) becomes, including the leading collisional correction term for *odd* modes,[10]

$$-\frac{en_i}{T_i}qR_0\left[\hat{\Pi}_0-\frac{\omega\omega_{*i}[1-\eta_i]}{(\pi\bar{\omega}_{ti})^2\epsilon_0^{1/2}}\hat{\Pi}_1-i\frac{\omega^2\omega_{*i}[1-\frac{3}{2}\eta_i]}{\pi^{5/2}\bar{\omega}_{ti}^3\epsilon_0}\hat{\Pi}_2-i\left(\frac{\nu_{iI}}{\epsilon_0}\right)^{1/2}\frac{\omega_{*i}[1-\frac{3}{2}\eta_i]}{(\pi\bar{\omega}_{ti})^{3/2}\epsilon_0^{1/4}}\hat{\Pi}_4\right]\,, \tag{190}$$

where

$$\hat{\Pi}_4 = \frac{8}{3\pi^{1/2}\epsilon_0} \int_{-1/2}^{1/2} dS \left(\int_{1-\epsilon_0}^{1+\epsilon_0} d\Lambda\, L_b^{5/2}(\Lambda) \sum_{p=1}^{\infty} |\tilde{\Phi}^{(p)}(\Lambda, S)|^2 / p^{3/2} + \frac{1}{2^{5/2}} \int_{1-\epsilon_0-\delta_0}^{1-\epsilon_0} d\Lambda\, L_t^{5/2}(\Lambda) \sum_{p=-\infty}^{\infty} |\tilde{\Phi}^{(p)}(\Lambda, S)|^2 / |p+S|^{3/2} \right), \tag{191}$$

and the rest of the $\hat{\Pi}$'s are defined in Sec. V.

Referring to the ordering given by (175) and (182), and collecting the results (177), (178), and (190), we see that the important terms in the quadratic form of the quasi-neutrality condition (19) are now[10]

$$\left(\frac{T_i}{T_e} + \frac{n_i}{n_e}\right)\hat{\Pi}_0 + Z\frac{n_I}{n_e}\frac{\omega_{*I}}{\omega}\hat{\Pi}_0 + i\left(\frac{\nu_{iI}}{\epsilon_0}\right)^{1/2} \times \left(\tfrac{3}{2}\eta_i - 1\right)\frac{n_i}{n_e}\frac{\omega_{*i}}{(\pi\bar{\omega}_{ti})^{3/2}\epsilon_0^{1/4}}\hat{\Pi}_4 = 0, \tag{192}$$

from which we derive the real frequency

$$\omega_0 = -Z\omega_{*I}\frac{n_I}{n_e}\left(\frac{T_i}{T_e} + \frac{n_i}{n_e}\right)^{-1}, \tag{193}$$

and the growth rate

$$\gamma = Z\frac{n_I n_i}{n_e^2}\omega_{*I}\omega_{*i}\left(\tfrac{3}{2}\eta_i - 1\right)\left(\frac{\nu_{iI}}{\epsilon_0}\right)^{1/2} \times \frac{1}{(\pi\bar{\omega}_{ti})^{3/2}\epsilon_0^{1/4}}\frac{1}{(T_i/T_e + n_i/n_e)^2}\frac{\hat{\Pi}_4}{\hat{\Pi}_0}. \tag{194}$$

We see that if $\eta_i > \frac{2}{3}$, the relevant instability is excited for $(d\ln n_I/dr)/(d\ln n_i/dr) > 0$, such as in the case where impurities may be concentrated at the center of the plasma column. Thus, we may argue that, for $\eta_i > \frac{2}{3}$, this instability should lead to a redistribution of the impurity concentration toward the outer edge of the plasma column. Clearly, a configuration in which $\eta_i > \frac{2}{3}$ and impurities are concentrated around the surface of the plasma column is stable to the considered type of mode.

We note that the impurity sound term obtained in Sec. V can be included in this analysis. The quadratic form (192) becomes, neglecting the impurity ion temperature gradient,

$$\left(\frac{T_i}{T_e} + \frac{n_i}{n_e}\right) - Z^2\frac{n_I}{n_e}\frac{m_i}{m_I}\frac{\bar{\omega}_{ti}^2}{2\omega^2}J_3 + Z\frac{n_I}{n_e}\frac{\omega_{*I}}{\omega} + i\left(\frac{\nu_{iI}}{\epsilon_0}\right)^{1/2}\left(\tfrac{3}{2}\eta_i - 1\right)\frac{n_i}{n_e}\frac{\omega_{*i}}{\bar{\omega}_{ti}^{3/2}\epsilon_0^{1/4}}J_4 = 0, \tag{195}$$

where $J_4 \equiv \hat{\Pi}_4/(\pi^{3/2}\hat{\Pi}_0)$. From this expression we obtain, taking $\omega_0 > \gamma$,

$$\omega_0 \simeq Z\frac{n_I}{2n_e}\omega_{*I}\left(\frac{T_i}{T_e} + \frac{n_i}{n_e}\right)^{-1} \times \left\{-1 \pm \left[1 + 2\left(\frac{T_i}{T_e} + \frac{n_i}{n_e}\right)\frac{n_e}{n_I}\frac{\bar{\omega}_{ti}^2}{\omega_{*I}^2}\frac{m_i}{m_I}J_3\right]^{1/2}\right\}, \tag{196}$$

and

$$\gamma \simeq \left(\frac{\nu_{iI}}{\epsilon_0}\right)^{1/2}\left(\tfrac{3}{2}\eta_i - 1\right)\frac{n_i}{n_I Z} \times \frac{\omega_0^2 J_4}{\sigma_I \bar{\omega}_{ti}^{3/2}\epsilon_0^{1/4}}\left(1 - Z\frac{m_i}{m_I}\frac{\bar{\omega}_{ti}^2}{\omega_0\omega_{*I}}J_3\right)^{-1}. \tag{197}$$

In order to examine a limit in which these results simplify, we require that the last term in (196) be small compared with one, which requires that

$$Z^2\frac{n_I}{n_e}\frac{m_i}{m_I} < \epsilon_0; \tag{198}$$

then the two roots for ω_0 are given approximately by (193) and by

$$\omega_0 \simeq Z\frac{\bar{\omega}_{ti}^2}{2\omega_{*I}}\frac{m_i}{m_I}J_3. \tag{199}$$

For the root (193), the last factor in (197) is positive definite, so that the instability conditions are just those discussed previously. For the root (199), the last factor in (197) is negative definite, so that the instability condition is the complement of the one for (193). Thus, one of the two roots is always unstable, as long as the condition (198) is satisfied.

The impurity modes considered so far in this section are all odd modes. Even modes, such as the ones described in Ref. 12, can also arise specifically because of the presence of impurities in appropriate collisionality regimes. Furthermore, the presence of impurities can modify modes which would be unstable in the absence of impurities. The effect can be either stabilizing or destabilizing, depending on the circumstances, as discussed in Refs. 12, 13, and 10.

This effect of impurities is illustrated by the dissipative trapped ion mode,[5] which is even and requires

$$\left(\frac{\nu_i}{\epsilon_0}\right) < \omega < \langle\omega_b\rangle_i, \quad \left(\frac{\nu_e}{\epsilon_0}\right) < \langle\omega_b\rangle_e. \tag{200}$$

Including the $p = 0$ terms for trapped particles in proper fashion, the dominant terms in the quadratic form are[10]

$$0 = \left(1 + \frac{T_i n_e}{T_e n_i}\right)\hat{\Pi}_0 + \frac{\omega_{*i}}{\omega}\epsilon_0^{1/2}\hat{\Pi}_5 + Z\frac{n_I}{n_i}\frac{\omega_{*I}}{\omega}\hat{\Pi}_0 - i\left(\frac{\nu_i}{\epsilon_0}\right)\frac{\omega_{*i}}{\omega^2}\left(1 - \tfrac{3}{2}\eta_i\right)\epsilon_0^{1/2}\left(\frac{2\hat{\Pi}_5}{3\pi^{1/2}}\right) - i\frac{\omega_{*e}\epsilon_0^{1/2}}{(\nu_e/\epsilon_0)}\left(1 + \tfrac{3}{2}\eta_e\right)\frac{T_i n_e}{T_e n_i}\left(\frac{4\hat{\Pi}_5}{\pi^{1/2}}\right), \tag{201}$$

where

$$\hat{\Pi}_5 = \frac{1}{2\epsilon_0}\int_{-1/2}^{1/2} dS \int_{1-\epsilon_0}^{1+\epsilon_0} d\Lambda\, L_b(\Lambda)\, |\tilde{\Phi}^{(0)}(\Lambda, S)|^2. \tag{202}$$

This quadratic form may be solved approximately to yield

$$\omega_0 \simeq \frac{\omega_{*e}}{(n_e/n_i + T_e/T_i)}\left(\epsilon_0^{1/2}\frac{\hat{\Pi}_5}{\hat{\Pi}_0} + Z\frac{dn_I/dr}{dn_i/dr}\right), \tag{203}$$

and

$$\gamma \simeq \frac{4/\pi^{1/2}}{\left(1 + Z\epsilon_0^{-1/2}\dfrac{dn_I/dr}{dn_i/dr}\dfrac{\hat{\Pi}_0}{\hat{\Pi}_5}\right)} \times \left[\frac{\omega_0^2}{(\nu_e/\epsilon_0)}\frac{n_e}{n_i}\left(1 + \tfrac{3}{2}\eta_e\right) + \frac{1}{6}\left(\frac{\nu_i}{\epsilon_0}\right)\left(\tfrac{3}{2}\eta_i - 1\right)\right]. \tag{204}$$

Thus, we see that when the impurity density gradient is of contrary sign to that of the main ion population, the dissipative trapped ion mode is stabilized if

$$\eta_i > \frac{2}{3} - \frac{\omega_0^2 \epsilon_0^2}{\nu_i \nu_e} \frac{4n_e}{n_i}(1 + \tfrac{3}{2}\eta_e) , \tag{205}$$

and

$$\sigma_I < -\frac{\hat{\Pi}_5}{\hat{\Pi}_0} \frac{\epsilon_0^{1/2} Z}{Z_e} , \tag{206}$$

or if both inequalities are reversed. Condition (206) is somewhat extreme and is unlikely to be satisfied. It is, in fact, reasonable to assume that the contribution of impurities in the expression (204) of the growth rate is negligible under a variety of realistic conditions.[13]

VII. COLLISIONAL EQUILIBRIA WITH IMPURITIES

In order to assess the effects of the anomalous particle and thermal energy transport due to impurity modes, it is important to recall the dependence of the collisional impurity flux on the density and temperature gradients of the ion species involved. The corresponding equilibrium density profiles are essentially determined by the requirement that the impurity flux vanish, at least to lowest order in $(m_e/m_i)^{1/2}$. To illustrate the procedures involved, we consider the regime in which both the main ions and impurity ions are collision dominated, and at first, we refer only to the case of one-dimensional geometry, ignoring toroidal effects. The stationary forms of the transverse momentum conservation equations are[7]

$$0 = -\nabla_\perp p_i + e n_i(\mathbf{E} + c^{-1}\mathbf{u}_i \times \mathbf{B})_\perp - \nu_{iI} m_i n_i (\mathbf{u}_i - \mathbf{u}_I)_\perp \tag{207}$$

for the main ions, and

$$0 = -\nabla_\perp p_I + Zen_I(\mathbf{E} + c^{-1}\mathbf{u}_I \times \mathbf{B})_\perp + \nu_{iI} m_i n_i (\mathbf{u}_i - \mathbf{u}_I)_\perp \tag{208}$$

for the impurity ions. Then we have, recalling that we consider the limit $\Omega_i/\nu_{iI} \gg 1$,

$$\mathbf{u}_{I\perp} - \mathbf{u}_{i\perp} \simeq \frac{c}{eB}\left(\frac{1}{n_i}\nabla p_i - \frac{1}{Zn_I}\nabla p_I\right) \times \frac{\mathbf{B}}{B} + \frac{\nu_{iI}}{m_i \Omega_i^2}\left(1 + \frac{n_i}{Zn_I}\right)\left(\frac{1}{n_i}\nabla p_i - \frac{1}{Zn_I}\nabla p_I\right) . \tag{209}$$

The collisional transport of impurities is directed toward the center of the plasma column if

$$\frac{1}{n_i}\frac{dp_i}{dr} - \frac{1}{Zn_I}\frac{dp_I}{dr} < 0 ,$$

a situation that is realized, for instance, when impurities are concentrated at the outer edge of the plasma column. The characteristic time scale for this process is

$$\tau_D \sim \frac{Zn_I}{n_i} \frac{r_{ni} r_{nI}}{\rho_i^2} \frac{1}{\nu_{iI}} , \tag{210}$$

where a is the plasma column radius and ρ_i is the ion gyroradius, and is typically much longer than the growth times of the impurity driven modes discussed in Secs. IV, V, and VI. For the collision-dominated impurity modes of Sec. III this may or may not be true. The relevant flow vanishes when impurities have sufficiently accumulated at the center of the plasma column so that

$$\sigma_I \equiv \frac{d\ln n_I/dr}{d\ln n_i/dr} = Z\frac{T_i}{T_I}(1 + \eta_i) - \frac{d\ln T_I/dr}{d\ln n_i/dr} . \tag{211}$$

Referring to toroidal confinement configurations, the collisional transport depends on the relative value of the effective collision frequency of trapped ions to their bounce frequency. In particular, three regimes have been analyzed[8,14,15]

$$\text{(a)}\quad \nu_i/\epsilon_0 > \langle\omega_b\rangle_i , \quad \nu_I/\epsilon_0 > \langle\omega_b\rangle_I , \tag{212}$$

$$\text{(b)}\quad \nu_i/\epsilon_0 < \langle\omega_b\rangle_i , \quad \nu_I/\epsilon_0 < \langle\omega_b\rangle_I , \tag{213}$$

$$\text{(c)}\quad \nu_i/\epsilon_0 < \langle\omega_b\rangle_i , \quad \nu_I/\epsilon_0 > \langle\omega_b\rangle_I . \tag{214}$$

In the latter two cases the condition for vanishing impurity flux acquires a new term in comparison to (211), that can reverse the sign of σ_I. In particular, if the contribution of $d\ln T_I/d\ln n_i$ is neglected, this condition can be written as

$$\sigma_I = Z\frac{T_i}{T_I}\{1 - \eta_i[d(Z_e) - 1]\} , \tag{215}$$

where $Z_e = Z^2 n_I/n_i$ and the function $d(Z_e)$ depends on the specific regime considered. Thus, in the case of regime (a),[8]

$$d(Z_e) \simeq \left(0.30 + \frac{0.41}{0.58 + Z_e}\right)\left(0.47 + \frac{0.35}{0.66 + Z_e}\right)^{-1} .$$

It should be noticed that $d(Z_e)$ approaches unity as $Z_e \to 0$, and is less than one for all finite values of Z_e for this case. For regime (b),[14]

$$d(Z_e) \simeq 1 + \frac{0.09 + 0.50\,Z_e}{0.53 + Z_e} ,$$

while for regime (c),[15]

$$d(Z_e) \simeq \tfrac{3}{2} .$$

If we define

$$\eta_c \equiv \frac{1}{d(Z_e) - 1} , \tag{216}$$

the relevant flow of impurities is inward for $\eta_i < \eta_c$, and they will tend to accumulate at the center of the plasma column. For $\eta_i > \eta_c$, the relevant flow is outward and the impurities will tend to concentrate at the edge of the plasma column. We should, of course, consider that η_i and η_c are functions of the plasma radius so that it is difficult to indicate, in general, the precise type of impurity density profile that can be generated. For the cases treated in Refs. 14 and 15, the value of η_c is always greater than or about two.

As we pointed out in the previous sections, the direction of the impurity flows produced by the modes we have discussed depends on the values of η_i/η_{ic}. In particular, $\eta_{ic} = 1 \div 2$ for the short wavelength collisionless modes treated in Sec. IV and $\eta_{ic} = \frac{2}{3}$ for the collisional modes treated in Sec. III as well as for the "trapped-ion" modes treated in Secs. V and VI. We notice that a condition where $\eta_i > \eta_{ic}$ can be more realistically realized in actual experiments than $\eta_i \gtrsim 2$ as required by the collisional equilibrium theory for an outward-directed impurity flow. Thus, we may imagine situations where the quasi-linear effects of impurity modes oppose the direct effect of collisions, which is to bring impurities into the plasma, when η_i is greater than η_{ic} for impurity modes, but less than η_c for collisions.

Finally, we summarize the properties of the modes we have discussed in Table I.

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APPENDIX

In the following we shall derive the full dispersion relation of the dissipative mode treated in Sec. III.

The linearized main ion and impurity ion momentum balance equations along the magnetic field are

$$m_i n_i\left[\frac{\partial}{\partial t} - \nabla_\parallel\left(\frac{\mu_i T_i}{m_i \nu_i}\right)\nabla_\parallel\right]\tilde{u}_{i\parallel} = -[n_i e\nabla_\parallel\tilde{\phi} + T_i\nabla_\parallel\tilde{n}_i + (1+\alpha_{iI})n_i\nabla_\parallel\tilde{T}_i] - m_i n_i \beta_{iI}\nu_{iI}(\tilde{u}_{i\parallel} - \tilde{u}_{I\parallel}) ,$$

and

$$m_I n_I \frac{\partial}{\partial t} u_{I\parallel} = -n_I Ze\nabla_\parallel\tilde{\phi} - n_I\nabla_\parallel\tilde{T}_I - T_I\nabla_\parallel\tilde{n}_I + \alpha_{iI} n_i \nabla_\parallel\tilde{T}_i + \beta_{iI}\nu_{iI} m_i n_i(\tilde{u}_{i\parallel} - \tilde{u}_{I\parallel}) ,$$

which reduce, respectively, to

$$\left(1 + i\mu_i\frac{k_\parallel^2 T_i}{\omega\nu_i m_i}\right)\tilde{u}_{i\parallel} = \frac{k_\parallel}{m_i n_i \omega}[n_i e\tilde{\phi} + T_i\tilde{n}_i + n_i(1+\alpha_{iI})\tilde{T}_i] - i\beta_{iI}\frac{\nu_{iI}}{\omega}(\tilde{u}_{i\parallel} - \tilde{u}_{I\parallel}) , \tag{A1}$$

and

$$u_{I\parallel} = \frac{k_\parallel}{\omega m_I n_I}(n_I Ze\tilde{\phi} + \tilde{n}_I T_I + n_I\tilde{T}_I - \alpha_{iI} n_i\tilde{T}_i) + i\beta_{iI}\frac{m_i n_i}{m_I n_I}\frac{\nu_{iI}}{\omega}(\tilde{u}_{i\parallel} - \tilde{u}_{I\parallel}) , \tag{A2}$$

where μ_i is a viscosity coefficient. The main ion and impurity ion thermal energy balance equations are

$$\left[\frac{3}{2}\frac{\partial}{\partial t} - \nabla_\parallel\left(\frac{\chi_i T_i}{\nu_i m_i}\nabla_\parallel\right)\right]\tilde{T}_i = \tfrac{3}{2}c\frac{\nabla\tilde{\phi}\times\mathbf{B}}{B^2}\cdot\nabla T_i - T_i\nabla_\parallel(\tilde{u}_{i\parallel}) - \alpha_{iI}T_i\nabla_\parallel(\tilde{u}_{i\parallel} - \tilde{u}_{I\parallel}) - 3\frac{m_i}{m_I}\nu_{iI}(\tilde{T}_i - \tilde{T}_I) ,$$

and

$$\frac{3}{2}\frac{\partial}{\partial t}\tilde{T}_I = \tfrac{3}{2}c\frac{\nabla\tilde{\phi}\times\mathbf{B}}{B^2}\cdot\nabla T_I - T_I\nabla_\parallel(\tilde{u}_{I\parallel}) + 3\frac{m_i n_i}{m_I n_I}\nu_{iI}(\tilde{T}_i - \tilde{T}_I) ,$$

which can be written, respectively, as

$$\left(\frac{3}{2} + i\chi_i\frac{k_\parallel^2 T_i}{\omega\nu_i m_i}\right)\frac{\tilde{T}_i}{T_i} = -\frac{3}{2}\frac{e\tilde{\phi}}{T_i}\frac{\omega_{Ti}}{\omega} + \frac{k_\parallel\tilde{u}_{i\parallel}}{\omega} + \alpha_{iI}\frac{k_\parallel(\tilde{u}_{i\parallel} - \tilde{u}_{I\parallel})}{\omega} - 3i\frac{m_i}{m_I}\frac{\nu_{iI}}{\omega}\frac{\tilde{T}_i - \tilde{T}_I}{T_i} , \tag{A3}$$

and

$$\frac{3}{2}\frac{\tilde{T}_I}{T_i} = -\frac{3}{2}\frac{e\tilde{\phi}}{T_i}\frac{\omega_{TI}}{\omega} + \frac{k_\parallel\tilde{u}_{I\parallel}}{\omega}\frac{T_I}{T_i} + 3i\frac{m_i n_i}{m_I n_I}\frac{\nu_{iI}}{\omega}\frac{\tilde{T}_i - \tilde{T}_I}{T_i} , \tag{A4}$$

where $\omega_{Tj} \equiv k_y(c/eB)(dT_j/dx)$ with $j = i, I$.

We consider the limit in which impurity-ion collisions are dominant [see Eq. (28)], so that

$$\frac{m_i n_i}{m_I n_I}\frac{\nu_{iI}}{\omega} > 1 .$$

Then, we find from Eq. (A4) that the energy transfer due to $i - I$ collisions is such that

$$\tilde{T}_I \simeq \tilde{T}_i .$$

Since we take $n_I/n_i \ll 1$, we may neglect the collisional energy transfer to the main ions. Then, the main ion thermal energy balance equation (A3) can be written as

$$\left(\frac{3}{2} + i\chi_i\frac{k_\parallel^2 T_i}{\omega\nu_i m_i}\right)\frac{\tilde{T}_i}{T_i} = +\frac{e\tilde{\phi}}{T_i}\frac{\omega_{*i}}{\omega}[1 - \tfrac{3}{2}\eta_i) + \frac{\tilde{n}_i}{n_i} + \alpha_{iI}\frac{k_\parallel}{\omega}(\tilde{u}_{i\parallel} - \tilde{u}_{I\parallel}) , \tag{A5}$$

where we made use of the main ion mass conservation (23) to eliminate $\tilde{u}_{i\parallel}$.

The two ion momentum balance Eqs. (A1) and (A2) reduce to

$$\frac{n_e}{n_i}\frac{e\tilde{\phi}}{T_i} + \frac{\tilde{n}_i}{n_i} + \frac{\tilde{T}_i}{T_i} + \frac{T_I}{T_i}\frac{\tilde{n}_I}{n_i} = 0 , \tag{A6}$$

and

$$\frac{k_\parallel}{\omega}(\tilde{u}_{i\parallel} - \tilde{u}_{I\parallel}) = \frac{i}{\beta_{iI}}\frac{k_\parallel^2 T_i}{\omega\nu_{iI} m_i}\left(\frac{Z_e}{Z}\frac{e\tilde{\phi}}{T_i} + \frac{T_I}{T_i}\frac{\tilde{n}_I}{n_i} - \alpha_{iI}\frac{\tilde{T}_i}{T_i}\right) , \tag{A7}$$

where we have neglected contributions of order n_I/n_i. Subtracting the main ion and impurity ion mass conservation Eq. (23) gives

$$\frac{\tilde{n}_I}{n_I} = \frac{\tilde{n}_i}{n_i} + \frac{\omega_{*i}}{\omega}(1 - \sigma_I)\frac{e\tilde{\phi}}{T_i} - \frac{k_\parallel}{\omega}(\tilde{u}_{i\parallel} - \tilde{u}_{I\parallel}) . \tag{A8}$$

From Eqs. (A5)–(A8) and the quasi-neutrality condition

$$\frac{\tilde{n}_I}{n_I} = \frac{Z}{Z_e}\left(\frac{n_e}{n_i}\frac{e\tilde{\phi}}{T_e} - \frac{\tilde{n}_i}{n_i}\right) , \tag{A9}$$

we obtain the dispersion relation

$$\hat{\omega} + A_1\frac{Z_e}{Z}[\sigma_I - 1 + \hat{\alpha}_I(\tfrac{3}{2}\eta_i - 1)] + i\frac{A_3}{Z^2\hat{C}} - i\hat{\omega}\hat{C}\left[\hat{\omega} + A_2\left(\frac{5}{2}\frac{Z_e}{Z}(\sigma_I - 1) - (\tfrac{3}{2}\eta_i - 1)\right)\right] = 0 , \tag{A10}$$

where

$$\hat{\omega} = \frac{\omega}{\omega_{*i}} , \quad \hat{\chi}_i = \chi_i + \frac{\alpha_{iI}^2}{\beta_{iI}}\frac{\nu_i}{\nu_{iI}} , \quad \hat{\alpha}_I = \frac{\alpha_{iI}\nu_i}{\beta_{iI}\hat{\chi}_i\nu_{iI}} ,$$

$$A_1 = \left(1 + \frac{T_i}{T_e}\right)^{-1} , \quad A_2 = \left(\frac{3}{2} + \frac{5}{2}\frac{T_i}{T_e}\right)^{-1} ,$$

$$A_3 = A_1^2 A_2^{-1} Z_e\frac{\nu_i\chi_i}{\nu_{iI}\beta_{iI}\hat{\chi}_i^2}\left[Z_e + \frac{T_I}{T_i}\left(1 + \frac{T_i}{T_e}\right)\right] ,$$

$$\hat{C}^{-1} = A_1^{-1}A_2\hat{\chi}_i\frac{k_\parallel^2 T_i}{\omega_{*i}\nu_i m_i} . \tag{A11}$$

In deriving Eq. (A10) we have neglected terms of order

Z^{-1}. Equation (A10) may be solved for $\hat{\omega} = \hat{\omega}_0 + i\hat{\gamma}$ to yield

$$\hat{\omega} = -\left(\frac{Z_e}{Z} A_1[(\sigma_I - 1) + \hat{\alpha}_I(\tfrac{3}{2}\eta_i - 1)] + i\,\frac{A_3}{Z^2\hat{C}}\right) \times \frac{1 + \hat{C}\hat{\gamma} + i\hat{C}[\hat{\omega}_0 + \frac{5}{2}A_2(Z_e/Z)(\sigma_I - 1) - A_2(\frac{3}{2}\eta_i - 1)]}{(1 + \hat{C}\hat{\gamma})^2 + \hat{C}^2[\hat{\omega}_0 + \frac{5}{2}A_2(Z_e/Z)(\sigma_I - 1) - A_2(\frac{3}{2}\eta_i - 1)]^2}\,, \tag{A12}$$

so that the condition for instability is

$$-\frac{Z_e}{Z}A_1[(\sigma_I - 1) + \hat{\alpha}_I(\tfrac{3}{2}\eta_i - 1)] \times [\hat{\omega}_0 + \tfrac{5}{2}(Z_e/Z)A_2(\sigma_I - 1) - A_2(\tfrac{3}{2}\eta_i - 1)] - \frac{A_3}{Z^2\hat{C}^2} > 0\,. \tag{A13}$$

We see that if the last term in Eq. (A13) is dominant, no instability can occur. In the limit where $k_\parallel^2 T_i/\omega\nu_i m_i > 1$, but is not too large, such that

$$\frac{1}{Z_e Z|\sigma_I|} < \hat{C}^2 < 1\,, \tag{A14}$$

we obtain

$$\hat{\omega}_0 = -(Z_e/Z)A_1[\sigma_I - 1 + \hat{\alpha}_I(\tfrac{3}{2}\eta_i - 1)]\,, \tag{A15}$$

and

$$\hat{\gamma} = -\hat{\omega}_0 A_2\hat{C}[\tfrac{3}{2}\eta_i - 1 - (Z_e/Z)A_1(\sigma_I - 1)]\,. \tag{A16}$$

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